

An easy and computable approximation for Troesch's problem by using the Laplace Transform-Homotopy Perturbation Method

Una solución aproximada y práctica al problema de Troesch utilizando el Método de Perturbación Homotópica con Transformada de Laplace

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Abstract

This work introduces the Laplace Transform-Homotopy Perturbation Method (LT-HPM) in order to provide an approximate solution for Troesch's problem. After comparing figures between exact and approximate solutions, as well as the average absolute relative error (AARE) of the approximate solutions of this research, with others reported in the literature, it can be said that the proposed solutions are accurate and handy. In conclusion, LT-HPM is a potentially useful tool.

Keywords: Laplace Transform; Homotopy Perturbation Method; Troesch's Problem.

Resumen

Este artículo propone el método de perturbación homotópica con Transformada de Laplace para encontrar una solución aproximada al problema de Troesch. Después de comparar las figuras entre la solución aproximada y la exacta, así como el error relativo absoluto promedio (AARE) de las soluciones aproximadas en este estudio, con otras reportadas en la literatura, se puede decir que la solución propuesta es, además de práctica, precisa. Se concluye que el método LTHPM es potencialmente útil.

Palabras clave: Transformada de Laplace; Método de Perturbación Homotópica; Problema de Troesch.

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Introducción

Troesch's equation is important in physics because it models the confinement of a plasma column by radiation pressure. Thus, it is relevant to search for precise solutions for this equation. Unfortunately, Troesch's equation is difficult to solve, as it happens with several nonlinear differential equations that appear in the physical sciences.

Laplace transform (LT) has played a relevant role, not only for its theoretical interest but also for its methods that allow to solve, in a simpler way, ordinary differential equations (ODE) that model many problems in the fields of engineering and science, in comparison with other mathematical methods (Spiegel, 1988). Specifically, LT has been useful for the solution of ODE with constant coefficients and initial conditions; also, LT can be employed in the case of certain differential equations with variable coefficients and partial differential equations (Spiegel, 1988). The applications of LT for nonlinear ODES has been focused, mainly, on getting analytical approximate solutions. Thus, Aminikhah & Hemmatnezhad (2012) reported LT-HPM, which is a coupling of the homotopy perturbation method (HPM) and LT methods aiming to get precise approximate solutions for these equations. Nevertheless, just as it happens with LT, LT-HPM has been employed, above all, to solve problems with initial conditions (Aminikhah, 2012a; Aminikhah & Hemmatnezhad, 2012), because LT-HPM is directly related to these problems. This work presents LT-HPM with the goal of finding an approximate solution for the nonlinear ODE that describes Troesch's problem, which is defined on a finite interval with Dirichlet boundary conditions.

ODE with boundary conditions on infinite intervals has been regarded in some papers, and it often corresponds to problems defined on semi-infinite ranges (Aminikhah, 2012b; Khan, Gondal, Hussain & Vanani, 2011); nevertheless, the way to solve these problems is different from the method presented in this work. The importance of research on nonlinear ODE is that many phenomena, theoretical and practical, have a nonlinear nature. Recently, various methods, alternative to classical methods, have risen to search approximate solutions for nonlinear differential equations, such as variational approaches (Assas, 2007; He, 2007; Kazemnia, Zahedi, Vaezi & Tolou, 2008; Noorzad, Poor & Omidvar, 2008), tanh method (Evans & Raslan, 2005), exp-function (Mahmoudi, Tolou, Khatami, Barari & Ganji, 2008; Xu, 2007), Adomian's decomposition method (ADM) (Adomian, 1988; Babolian & Biazar, 2002; Chowdhury, 2011; Deeba, Khuri & Xie, 2000; Kooch & Abadyan, 2011; 2012; Vanani, Heidari & Avaji, 2011), parameter expansion (Zhang & Xu, 2007), homotopy perturbation method (HPM) (Aminikhah, 2012a; 2012b; Aminikhah & Hemmatnezhad, 2012; Araghi & Rezapour, 2011; Araghi & Sotoodeh, 2012; Bayat, Bayat & Pakar, 2014; Bayat, Pakar & Emadi, 2013; Belendez *et al.*, 2009; Biazar & Aminikhah, 2009; Biazar & Eslami, 2012; Biazar & Ghanbari, 2012; Biazar & Ghazvini, 2009; El-Shaed, 2005; Feng, Mei & He, 2007; Fereidon, Rostamiyan, Akbarzade & Ganji, 2010; Filobello-Niño *et al.*, 2012a; 2012b; 2015; Ganji, Mirgolbabaei, Miansari & Miansari, 2008; He, 1999; 2000; 2006a; 2006b; Khan *et al.*, 2011; Khan & Wu, 2011; Marinca & Herisanu, 2011; Mirmoradia, Hosseinpoura, Ghanbarpour & Barari, 2009; Madani, Fathizadeh, Khan & Yildirim, 2011; Rashidi, Pour, Hayat & Obaidat, 2012; Rashidi, Rastegari, Asadi & Beg, 2012; Sharma & Methi, 2011; Vazquez-Leal *et al.*, 2012; Vazquez-Leal *et al.*, 2013; Vazquez-Leal *et al.*, 2014; Vazquez-Leal, Castañeda-Sheissa, Filobello-Niño, Sarmiento-Reyes & Sanchez-Orea, 2012; Vazquez-Leal, Filobello-Niño, Castañeda-Sheissa, Hernandez-Martinez & Sarmiento-Reyes, 2012; Vazquez-Leal, Sarmiento-Reyes, Khan, Filobello-Nino & Diaz-Sanchez, 2012), homotopy analysis method (HAM) (Hassana & El-Tawil, 2011; Patel, Mehta & Pradhan, 2012), and perturbation method (PM) (Filobello-Niño *et al.*, 2013a), among others. In the same way, some exact solutions to nonlinear ODE have been introduced, for example, in Filobello-Niño *et al.* (2013b).

The organization of this work is as follows. Section Standard HPM introduces the basic idea of the Homotopy Perturbation Method. The next Section presents the LT-HPM method. In Case Study section, LT-

HPM is presented in order to find an approximate solution for Troesch's equation. Additionally, a discussion on the results is presented. Finally, a conclusions section presents the results obtained from this work.

Standard HPM

The HPM method was introduced with the aim to find approximate solutions to various kinds of nonlinear problems. Indeed, HPM is a combination of the classical perturbative technique and the homotopy, whose origin is found in the topology, but it is not restricted to small parameters. Besides, HPM often requires a few iterations to find accurate approximate solutions (He, 2000; 1999).

To understand how the HPM method works, consider a general nonlinear differential equation (He, 2000) expressed as:

$$A(U) - f(r) = 0, \quad r \in \Omega, \quad (1)$$

with boundary conditions

$$B(U, \partial U / \partial n) = 0, \quad r \in \Gamma, \quad (2)$$

Where A is a differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the domain boundary for Ω . Besides, (1) can be expressed as:

$$L(U) + N(U) - f(r) = 0, \quad (3)$$

where L is linear and N nonlinear.

In general, a homotopy is constructed as follows (He, 1999; 2000):

$$H(U, p) = (1-p)[L(U) - L(u_0)] + p[L(U) + N(U) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega. \quad (4)$$

or

$$H(U, p) = L(U) - L(u_0) + p[L(u_0) + N(U) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega \quad (5)$$

Where p is called the homotopy parameter, while u_0 is the first approximation for the solution of (3) that satisfies the boundary conditions.

The solution for (4) or (5) is expressed as:

$$U = v_0 + v_1 p + v_2 p^2 + \dots \quad (6)$$

By substituting (6) into (5) and comparing identical powers of p terms, there can be found values for v_0, v_1, v_2, \dots

After considering $p \rightarrow 1$, the following approximate solution for (1) is obtained:

$$U = v_0 + v_1 + v_2 + v_3 \dots \quad (7)$$

Basic Idea of Laplace Transform Homotopy Perturbation Method (LT-HPM)

The aim of this section is to show how LT-HPM is used with the purpose of obtaining approximate solutions for ODE, as (3) (Aminikhah, 2012b; Filobello-Nino et al., 2015).

The procedure for LT-HPM follows the same steps as HPM, but only until (5); next, LT is applied on both sides of (5) to get:

$$\mathfrak{L}\{L(U) - L(u_0) + p[L(u_0) + N(U) - f(r)]\} = 0 \quad (8)$$

From Spiegel (1988), the following is obtained:

$$s^n \mathfrak{L}\{U\} - s^{n-1}U(0) - s^{n-2}U'(0) - \dots - U^{(n-1)}(0) = \mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\}, \quad (9)$$

or

$$\mathfrak{L}(U) = \left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \left(\frac{1}{s^n}\right)\mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\}. \quad (10)$$

Applying inverse Laplace transform to both sides of (10), the following is obtained:

$$U = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \left(\frac{1}{s^n}\right)\mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\}\right\} \quad (11)$$

Next, it is assumed that the solutions of (3) are expressed as:

$$U = \sum_{n=0}^{\infty} p^n v_n \quad (12)$$

so that substituting (12) into (11) be:

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \left(\frac{1}{s^n}\right)\mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(\sum_{n=0}^{\infty} p^n v_n) + f(r)]\}\right\}. \quad (13)$$

After comparing coefficients with the same power of p :

$$p^0 : v_0 = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \mathfrak{L}\{L(u_0)\}\right\}'$$

$$p^1 : v_1 = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\{\mathfrak{L}\{-N(v_0) - L(u_0) + f(r)\}\}\right\}'$$

$$p^2 : v_2 = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\mathfrak{L}\{-N(v_0, v_1)\}\right\}'$$

$$p^3 : v_3 = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\mathfrak{L}\{-N(v_0, v_1, v_2)\}\right\}' \quad (14)$$

...

$$p^j : v_j = \mathfrak{I}^{-1} \left\{ \left(\frac{1}{s^6} \right) \mathfrak{I} \left\{ -N(v_0, v_1, v_2, \dots, v_j) \right\} \right\},$$

...

Assuming that the initial approximation adopts the form $U(0) = u_0 = \alpha_0$, $U'(0) = \alpha_1, \dots, U^{(n-1)}(0) = \alpha_{n-1}$; then an approximate solution is expressed in terms of the following limit:

$$u = \lim_{p \rightarrow 1} U = v_0 + v_1 + v_2 + \dots \tag{15}$$

Case Study

Next, LT-HPM is used in order to get a handy analytical approximate solution for the nonlinear problem:

$$\frac{d^2 y(x)}{dx^2} = \varepsilon \operatorname{senh}(\varepsilon y), \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y(1) = 1 \tag{16}$$

where ε is a positive parameter (Troesch's parameter).

The terms to be identified are

$$L(y) = y''(x) \tag{17}$$

and

$$N(y) = -\varepsilon \operatorname{senh}(\varepsilon y), \tag{18}$$

where prime denotes differentiation respect to x .

Next, on the right side of (16), in terms of its Taylor series, the following is expressed:

$$y'' = \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6} + \frac{\varepsilon^6 y^5}{120} + \frac{\varepsilon^8 y^7}{5040} + \dots \tag{19}$$

In order to show the usefulness of the proposed method, an accurate approximate solution will be obtained, keeping only the first two terms on the right side of (19).

Next, the following homotopy is constructed (see 4):

$$(1-p)(y'' - y_0'') + p \left[y'' - \varepsilon^2 y - \frac{\varepsilon^4 y^3}{6} \right] = 0 \tag{20}$$

or

$$y'' = y_0'' + p \left[-y_0'' + \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6} \right]. \tag{21}$$

After applying LT:

$$\mathfrak{T}(y'') = \mathfrak{T}\left(y_0'' + p\left[-y_0'' + \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6}\right]\right). \quad (22)$$

Following Spiegel (1988), the (22) is rewritten as:

$$s^2 Y(s) - sy(0) - y'(0) = \mathfrak{T}\left(y_0'' + p\left[-y_0'' + \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6}\right]\right), \quad (23)$$

where $Y(s) = \mathfrak{T}(y(x))$ has been defined.

After using $y(0) = 0$, (23) is expressed as:

$$s^2 Y(s) - A = \mathfrak{T}\left(y_0'' + p\left[-y_0'' + \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6}\right]\right), \quad (24)$$

Where $A = y'(0)$

Solving for $Y(s)$ and applying Laplace inverse transform \mathfrak{T}^{-1} , the next is obtained:

$$y(x) = \mathfrak{T}^{-1}\left\{\frac{A}{s^2} + \frac{1}{s^2} \mathfrak{T}\left(y_0'' + p\left[-y_0'' + \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6}\right]\right)\right\}. \quad (25)$$

Next, the following series solution for $y(x)$ will be assumed as

$$y(x) = \sum_{n=0}^{\infty} p^n v_n. \quad (26)$$

It is noted that

$$v_0(x) = Ax \quad (27)$$

is the first approximation for the solution for (16), which satisfies the conditions $y(0) = 0$, $y'(0) = A$.

Thus, after substituting (26) into (25), the following is obtained:

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{T}^{-1}\left\{\frac{A}{s^2} + \frac{1}{s^2} \mathfrak{T}\left(y_0'' + p\left[-y_0'' + \varepsilon^2 (v_0 + p v_1 + p^2 v_2 + \dots) + \frac{\varepsilon^4 (v_0 + p v_1 + p^2 v_2 + \dots)^3}{6}\right]\right)\right\} \quad (28)$$

After comparing coefficients of identical powers of p terms:

$$p^0: v_0(x) = \mathfrak{T}^{-1}\left\{\frac{A}{s^2}\right\}, \quad (29)$$

$$p^1: v_1(x) = \mathfrak{T}^{-1}\left\{\left(\frac{\varepsilon^2}{s^2}\right) \mathfrak{T}\{v_0\} + \left(\frac{\varepsilon^4}{6s^2}\right) \mathfrak{T}\{v_0^3\}\right\}, \quad (30)$$

$$P^2 : v_2(x) = \mathfrak{S}^{-1} \left\{ \left(\frac{\varepsilon^2}{s^2} \right) \mathfrak{S}(v_1) + \left(\frac{\varepsilon^4}{2s^2} \right) \mathfrak{S}(v_1 v_0^2) \right\}, \quad (31)$$

...

After solving Laplace transforms (29), (30), and (31):

$$p^0 : v_0(x) = Ax, \quad (32)$$

$$p^1 : v_1(x) = \frac{\varepsilon^2 Ax^3}{6} + \frac{A^3 \varepsilon^4 x^5}{120}, \quad (33)$$

$$p^2 : v_2(x) = \frac{\varepsilon^4 A}{120} x^5 + \frac{11\varepsilon^6 A^3}{5040} x^7 + \frac{\varepsilon^8 A^5}{17280} x^9. \quad (34)$$

...

and so on.

By substituting (32)-(34) into (15), and considering the limit value $p \rightarrow 1$, the handy approximation is obtained:

$$y(x) = Ax + \frac{\varepsilon^2 A}{6} x^3 + \left(\frac{\varepsilon^4 A(1+A^2)}{120} \right) x^5 + \frac{11\varepsilon^6 A^3}{5040} x^7 + \frac{\varepsilon^8 A^5}{17280} x^9. \quad (35)$$

In order to calculate the value of A , it is necessary that (35) satisfies the boundary condition $y(1)=1$. This condition gives rise to an algebraic equation for the unknown A . Proposing as case studies $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$, the values

$$A = 0.9590503635 \quad (\varepsilon = 0.5), \quad (36)$$

$$A = 0.8456306348 \quad (\varepsilon = 1), \quad (37)$$

and

$$A = 0.5323243593 \quad (\varepsilon = 2) \quad (38)$$

are obtained, respectively.

By substituting (36), (37), and (38) into (35), the following expressions are obtained:

$$y(x) = 0.9590503635x + 0.03996043182x^3 + 0.0009589392735x^5 + 0.00003008197995x^7 + 1.834101508x10^{-7}x^9. \quad (39)$$

$$y(x) = 0.8456306348x + 0.1409384391x^3 + 0.01208611363x^5 + 0.001319788295x^7 + 0.00002502417688x^9. \quad (40)$$

$$y(x) = 0.5323243593x + 0.3548829062x^3 + 0.09108915995x^5 + 0.02107032055x^7 + 0.0006332539792x^9. \quad (41)$$

Discussion

This article employed LT-HPM in the search for a handy and precise analytical approximate solution for the nonlinear ODE defined with finite boundary conditions that describe the Troesch's problem. Since the proposed method is expressed in terms of initial conditions for a given problem (see (14), this procedure consisted of expressing the approximated solutions in terms of $A = y'(0)$. It is noted that in order to determine the value of A , the boundary condition $y(1) = 1$ has to be satisfied. This condition defines an algebraic equation for the unknown A .

Figure 1 shows the comparison between numerical and approximate solutions (39)-(41) for the cases $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$, respectively. It is also noted that curves are in good agreement; thus, the potentiality of the proposed method in the search for solutions for nonlinear ODE with finite boundary conditions is clear.

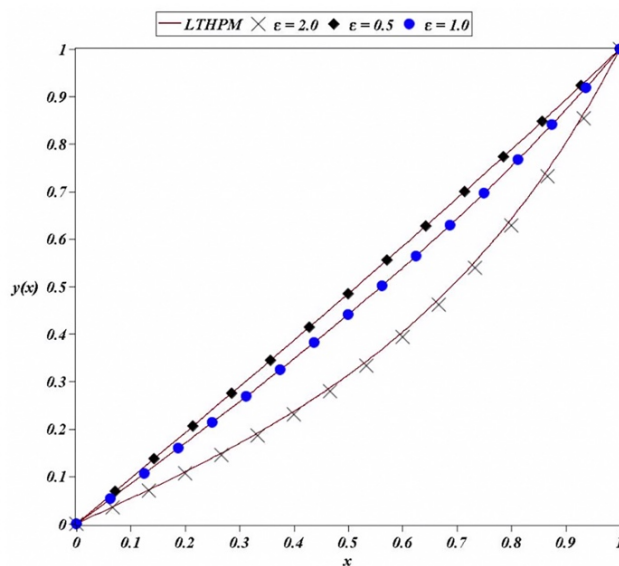


Figure 1. Comparison of proposed solution (35) (solid lines) for $\varepsilon = 1$, and $\varepsilon = 2$; versus the solutions reported in Erdogan & Ozis (2011) for $\varepsilon = 0.5$ (solid square), $\varepsilon = 1$ (solid circles), and $\varepsilon = 2$ (diagonal-cross) calculated by using the built-in numerical routine for BVP from Maple 15.

Source: Authors' own elaboration.

Table 1 shows the comparison between the exact solution reported by Erdogan & Ozis (2011), and approximations (39), ADM (Deeba *et al.*, 2000), HPM (Feng *et al.*, 2007), HPM (Mirmoradia *et al.*, 2009), and HAM (Hassana & El-Tawil, 2011) for the case $\varepsilon = 0.5$. It is clear that (39) is competitive with the second best accuracy; its average absolute relative error (AARE) is scarcely of 4.6×10^{-6} , only HAM (Hassana & El-Tawil, 2011) was more accurate, since its AARE is 2.51374×10^{-6} . Besides, HPM, ADM, and HAM methods are considered more difficult to use, because LT-HPM does not require solving recurrence differential equations. HAM usually requires more iterations and, consequently, more terms in comparison with LT-HPM; therefore, its expressions are many times long and cumbersome. Table 2 shows that, for $\varepsilon = 1$, approximation (40) possesses the lowest AARE 4.5×10^{-4} even though $\varepsilon = 1$ is not small (see below).

Table 1. Comparison between (39), the exact solution (Erdogan & Ozis, 2011), and other reported approximate solutions using $\varepsilon_{=0.5}$

x	Exact (Erdogan & Ozis, 2011)	This work, LT-HPM (39)	ADM (Deeba et al., 2000)	HPM (Feng et al., 2007)	HPM (Mirmoradia et al., 2009)	HAM (Hassana & El-Tawil, 2011)
0.1	0.0959443493	0.09594500637	0.0959383534	0.0959395656	0.095948026	0.0959446190
0.2	0.1921287477	0.1921300635	0.1921180592	0.1921193244	0.192135797	0.1921292845
0.3	0.2887944009	0.2887963775	0.2887803297	0.2887806940	0.288804238	0.2887952148
0.4	0.3861848464	0.3861874818	0.3861687095	0.3861675428	0.386196642	0.3861859313
0.5	0.4845471647	0.4845504381	0.4845302901	0.4845274183	0.4845599	0.4845485110
0.6	0.5841332484	0.5841370824	0.5841169798	0.5841127822	0.584145785	0.5841348222
0.7	0.6852011483	0.6852053362	0.6851868451	0.6851822495	0.685212297	0.6852028604
0.8	0.7880165227	0.7880205903	0.7880055691	0.7880018367	0.788025104	0.7880181729
0.9	0.8928542161	0.89285571853	0.8928480234	0.8928462193	0.892859085	0.8928553997
AARE		4.6×10^{-6}	3.47802×10^{-5}	3.57932×10^{-5}	2.44418×10^{-5}	2.51374×10^{-6}

Source: Authors' own elaboration.

Table 2. Comparison between (40), the exact solution (Erdogan & Ozis, 2011), and other reported approximate solutions, using $\varepsilon = 1$

x	Exact (Erdogan & Ozis, 2011)	This work, LT-HPM (40)	ADM (Deeba et al., 2000)	HPM (Feng, et al., 2007)	HPM (Mirmoradia, et al., 2009)	HAM (Hassana & El-Tawil, 2011)
0.1	0.0846612565	0.08470412291	0.084248760	0.0843817004	0.084934415	0.0846732692
0.2	0.1701713582	0.1702575190	0.169430700	0.1696207644	0.170697546	0.1701954538
0.3	0.2573939080	0.2575241867	0.256414500	0.2565929224	0.258133224	0.2574302342
0.4	0.3472228551	0.3473982447	0.346085720	0.3462107378	0.348116627	0.3472715981
0.5	0.4405998351	0.4408206731	0.439401985	0.4394422743	0.44157274	0.4406610140
0.6	0.5385343980	0.5387980978	0.537365700	0.5373300622	0.539498234	0.5386072529
0.7	0.6421286091	0.6424243421	0.641083800	0.6410104651	0.642987984	0.7526899495
0.8	0.7526080939	0.7529055047	0.751788000	0.7517335467	0.753267551	0.7526899495
0.9	0.8713625196	0.8715893682	0.870908700	0.8708835371	0.871733059	0.8714249118
AARE		4.5×10^{-4}	0.002714577	0.002320107	0.002044737	0.0019244326

Source: Authors' own elaboration.

An important feature from the proposed method is deduced from equations like (16), which are expressed in the following form $L(x) + \varepsilon N(x) = 0$, ($L(x)$ linear, $N(x)$ nonlinear). It is known that some methods as PM (Chow, 1995; Holmes, 1995) work better for values of $\varepsilon \ll 1$. For the problem presented here, reference is made to small values of Troesch's parameter. As a matter of fact, ε can be conceived as a parameter of smallness, which determines how greater the contribution of $L(x)$ is in comparison with $N(x)$. Generally, it is easier to find accurate analytical approximate solutions for small values of ε than for large values of this parameter. Figure 1 shows that (40) and (41) provide good approximations for the nonlinear problem (16), although perturbation parameters $\varepsilon = 1$ and $\varepsilon = 2$ are indeed large. Therefore, the proposed method is not limited to small parameters.

Finally, it is very important to emphasize that it is possible to improve the accuracy of the obtained approximations, adding higher order approximations to the solution (35) and keeping more terms in the Taylor series expansion (19).

Conclusions

In this article, LT-HPM was used to find a polynomial analytical approximate solution of five terms for the second order nonlinear ODE that describes Troesch's problem with Dirichlet boundary conditions defined on a finite interval. In general, the proposed method expresses the problem of obtaining an approximate solution for a nonlinear ordinary differential equation, in terms of solving an algebraic equation for some unknown initial condition. From figure 1, and table 1 and table 2, it is concluded that LT-HPM is a method with potential in the search for solutions for boundary value nonlinear problems. As it was already mentioned, an additional advantage of LT-HPM is that it does not require to solve several recurrence differential equations; therefore, it can be said that LT-HPM is a useful tool for practical applications.

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