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Bifurcation in calculus of variations with constraints

Bifurcación en el cálculo variacional con restricciones

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ABSTRACT

We describe a variational problem on a domain of a plane under a constraint of geometrical character. We provide sufficient and necessary conditions for the existence of bifurcation points. The problem in $\mathbb{R}^2$ coordinate form, corresponds to a quasilinear elliptic boundary value problem. The problem provides a physical model for several applications referring to continuum media and membranes.

RESUMEN

Se describe un problema variacional sobre una región del plano sometido a una restricción de carácter geométrico. Ofrecemos condiciones suficientes y necesarias para la existencia de puntos de bifurcación. El problema en las coordenadas de $\mathbb{R}^2$ corresponde a un problema elíptico semilineal en la frontera. El problema proporciona un modelo físico para varias aplicaciones relacionadas con los medios contínuos y membranas.

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INTRODUCTION

The classical bifurcation problem has the following form: Let X and Y be two Hilbert spaces. Consider the equation of the type

$$f(\lambda, u) = 0, \tag{1.1}$$

where

$$\frac{dN}{dt} = -\frac{\sigma_{12} r_w P}{\pi r_0^2}(N\xi_1 - 1) - \frac{N}{\tau} + P_{pump}$$

is a map dependent on a real parameter λ. We always assume that F is map sufficiently smooth with

$$f(\lambda, 0) = 0,$$

for all $\lambda \in \mathbb{R}$, *i.e.* $(\lambda, 0)$ is a trivial solution of the equation (1.1).

Definition 1.1. *The number $\lambda_0 \in \mathbb{R}$ is called a bifurcation point for the equation* (1.1) *if and only if, in every sufficiently small neighborhood of $(\lambda_0, 0)$ there exists a solution (u, λ) with $u \neq 0$.*

Proposition 1.1. *If λ_0 is a bifurcation point for the equation* (1.1) *then the derivative $f_u(\lambda_0, 0) : X \to Y$, is not invertible.*

This necessary condition for the existence of bifurcation points is a consequence of the implicit function theorem.

An interesting case is $X = Y$ and

$$f(\lambda, u) = \lambda u - g(u), \tag{1.2}$$

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under the assumption $g(0) = 0$. If λ_0 is a bifurcation point for the equation (1.2) then the derivative

$$f_u(\lambda_0, 0) = \lambda_0 I - g'(0),$$

is not invertible which means that λ_0 is an eigenvalue for the operator $g'(0)$. The question could be inverted: is every eigenvalue of $g'(0)$ a bifurcation point for the equation (1.2)? In general the answer is negative.

Example: Consider $X = Y = \mathbb{R}^2$,

$$g(x, y) = (x + y^3, y - x^3), \qquad f(\lambda; x, y) = \lambda(x, y) - g(x, y).$$

We obtain that $g'(0, 0) = I$ is the identity matrix, thus $\lambda_0 = 1$ is an eigenvalue of $g'(0, 0)$ but it is not a bifurcation point. If (x, y) is a solution, then

$$f(\lambda; x, y) = 0,$$

or equivalently

$$\lambda x = x + y^3,\ \lambda y = y - x^3,$$

or equivalently

$$x^4 + y^4 = 0,$$

and thus

$$(x, y) = (0, 0),$$

is the trivial solution. This means that there are no bifurcation points.

Bifurcation in calculus of variations

The bifurcation problem of variational character is of special interest since the integral functionals involved in this problem are models of the deformation energy of the continuum medium. The problem of the form (1.1) is reduced to

$$f(\lambda, u) = G'[u] - \lambda F'[u] = 0, \tag{2.1}$$

where G, F are functionals defined on the Hilbert space H with $F'[0] = G'[0] = 0$. It is proved by I. V. Skrypnik (1973) that if the functionals F, G satisfy some specific assumptions, then the necessary conditions for the existence of bifurcation points are also sufficient ones.

Let $\mathcal{V}$ be an open region of $u \in H$ and suppose that the functionals F and G satisfy the following conditions:

1. The functional F is weakly continuous, differentiable, and its differential is Lipschitz continuous with

$$F'[u] = Bu + L(u), \tag{2.2}$$

where B is a linear, bounded, self adjoint and positive definite operator. For the nonlinear part L, the following estimates hold:

$$\| L(u) \| \le c \| u \|^r,$$

$$\| L(u_1) - L(u_2) \| \le c\left(\| u_1 \|^{r-1} + \| u_2 \|^{r-1}\right)\| u_1 - u_2 \|, \tag{2.3}$$

where c is a positive constant, $r > 1$ and $u, u_1, u_2 \in \mathcal{V}$.

2. The functional G is weakly continuous, differentiable and its differential is Lipschitz continuous with

$$G'[u] = Au + N(u), \tag{2.4}$$

where A is a linear self adjoint and compact operator. For the nonlinear part N, the following estimate holds:

$$\| N(u) \| \le c \| u \|^p, \tag{2.5}$$

where c is a positive constant, $p > 1$ and $u \in \mathcal{V}$.

Proposition 2.1. *Every number $\lambda \in \mathbb{R}$, corresponding to a critical point $u \neq 0$ of the funtional*

$$I[\lambda, u] = G[u] - \lambda F[u],$$

is a bifurcation point of the equation

$$f(\lambda, u) = I_u[\lambda, u] = G'[u] - \lambda F'[u] = 0, \tag{2.6}$$

if and only if, the equation

$$f_u(\lambda, 0)u = I_{uu}[\lambda, 0]u = G''[0]u - \lambda F''[0]u = 0,\ u \in H \tag{2.7}$$

has a nontrivial solution.

Note that under these assumptions equation (2.7) can be rewritten as

$$Au - \lambda Bu = 0.$$

In addition to the bifurcation problem of variational character, the following problem attracted some special interest

$$f[u, \lambda] = G'[u] - \lambda F'[u] = 0, \quad \Phi[u] = 0, \tag{2.8}$$

where the differentiable mapping $\Phi : X \to \mathbb{R}$ satisfies $\Phi[0] = 0$. A problem of this type is called a bifurcation problem under the restriction of a constraint. The equation of the constraint

$$\Phi[u] = 0 \tag{2.9}$$

restricts the domain of (2.1) to a smaller subspace according to Lyapunov-Schmidt decomposition. We obtain

$$H = H_1 \oplus H_2, \quad \text{where} \quad H_1 = Ker\, \Phi'[0] \neq 0, \qquad H_2 = H_1^{\perp}.$$

Thus the solutions of the equation (2.9) for small values of $\| u \|$ can be represented as

$$u = v + h(v),\ v \in H_1, \tag{2.10}$$

where h is a differentiable map of a small region of $0 \in H_1$ to a small region of $0 \in H_2$ with

$$h(0) = 0,\ h'(0) = 0.$$

According to (2.10) we define the functionals:

$$J[v] = G[v + h(v)] = G[u], \qquad v \in H_1, \tag{2.11}$$

and

$$Q[v] = F[v + h(v)] = F[u], \qquad v \in H_1. \tag{2.12}$$

Then the derivatives

$$D_v F[u] = Q'[v], \qquad D_v G[u] = J'[v],$$

have the meaning of differentiation of the functionals F and G along the tangential direction of the manifold $\{v + h(v),\ v \in H_1\}$. Thus, the bifurcation problem (2.8) is equivalent to a problem

$$Q'[v] - \lambda J'[v] = 0, \qquad v \in H_1, \tag{2.13}$$

or equivalently

$$f(\lambda, u) = D_v F[u] - \lambda D_v G[u] = 0, \qquad u \in H. \tag{2.14}$$

Definition 2.1. *The number* $\lambda_0 \in \mathbb{R}$ *is a bifurcation point for equation* (2.14) *if in the intersection of a sufficient small neighborhood* $\mathcal{U} \subset X \times \mathbb{R}$ *of* $(0, \lambda_0)$ *with the manifold* $\{v + h(v),\ v \in H_1\}$ *there exists a nonzero solution of equation* (2.14).

Suppose that the functionals $F[u]$, $G[u]$, where $u \in H$ satisfy the conditions (2.2), (2.3), (2.4), (2.5). It can be proved that the functionals $Q[v]$, $J[v]$, where $v \in H_1$ satisfy the same conditions with $r \geq 2$, as well as the appropriates conditions of continuity and differentiability. In addition suppose that the functional $\Phi[u]$ is differentiable with $\Phi[0] = 0$ and $\mathrm{Ker}\Phi'[0] = H_1 \neq 0$.

Proposition 2.2. *The number* λ_0 *is a bifurcation point of the problem* (2.8) *or* (2.14) *if and only if, the equation*

$$PAP\,u - \lambda_0 PBP u = 0, \qquad u \in H,$$

where $P : H \to H_1$ *is the orthogonal projector, has a nontrivial solution.*

It is obvious that bifurcation points exists when $PAP \neq 0$.

Applications

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary sufficiently smooth consisting of two non-intersecting sufficiently smooth components Γ and Γ_1: $\partial\Omega = \Gamma \cup \Gamma_1$ and $\Gamma \cap \Gamma_1 = \emptyset$. For a funtional $u \in H$ where

$$H = \{\vec{u} \in W_2^1(\Omega, \mathbb{R}^2), \qquad \vec{u}\,|_{\Gamma_1} = 0, \qquad \vec{u}\,|_{\Gamma} \in W_2^2(\Gamma, \mathbb{R}^2)\},$$

are a Hilbert space, we introduce the functionals

$$F[\vec{u}] = \frac{1}{2}\int_\Omega \alpha_{ijkl}(x)\xi_{ij}(\vec{u})\xi_{kl}(\vec{u})dx + \frac{1}{2}\int_\Gamma |\,\delta_i\delta_i\vec{u}\,|^2\,ds,$$

$$G[\vec{u}] = \int_\Gamma q(\vec{u}, x)ds,$$

$$I[\lambda, \vec{u}] = F[\vec{u}] - \lambda G[\vec{u}], \quad \lambda \in \mathbb{R}, \tag{3.1}$$

where the coefficients $a_{ijkl} \in L_\infty(S)$ satisfy the symmetry property

$$a_{ijkl}(x) = a_{klij}(x) = a_{jikl}(x) = a_{ijlk}(x),$$

and they are positive definite, *i.e.*

$$a_{ijkl}(x)\xi^{ij}\xi^{kl} \geq \Lambda\xi^{ij}\xi^{ij}, \qquad \Lambda > 0,$$

for all the matrixes $(\xi^{ij})_{(i,j)}$. The tensor $\xi_{ij}(\vec{u})$ is defined as

$$\xi_{ij}(\vec{u}) = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right),$$

and δ_i is the tangential gradient

$$\delta_i = \frac{\partial}{\partial x^i} - \eta^i(x)\eta^j(x)\frac{\partial}{\partial x^j}, \qquad i = 1, 2, \qquad x \in \Gamma,$$

where $\vec{n}(x)$ is the unitary normal vector to Γ. Finally, suppose that

$$q_{u^i}(\vec{0}, x) = 0, \qquad i = 1, 2, \qquad x \in \Gamma.$$

The physical interpretation is that the functional $I[\vec{u}, \lambda]$ can be considered as the energy functional of a continuum medium with special characteristics determined by the coefficients a_{ijkl}. The medium is the interior of a shell Γ, which is under the influence of a force density coming from a potential $\lambda q(\vec{u}, x)$. The medium is fixed up to a part Γ_1 of the boundary ∂S. Hence, the first term of the functional $F[\vec{u}]$ represents the random deformation of the medium while the rest of the expression comes from the deformation of the shell.

The critical points $\vec{u} \in H$ of the energy functional $I[\lambda, \vec{u}]$ satisfy the integral equation

$$\int_\Omega \alpha_{ijkl}(x)\xi_{ij}(\vec{u})\xi_{kl}(\vec{v})dx + \int_\Gamma \delta_i\delta_i\vec{u}\delta_j\delta_j\vec{v}ds$$

$$-\lambda\int_\Gamma q_{u^i}(\vec{u},x)v^i\,ds = 0 \qquad (3.2)$$

for all $\vec{v}\in H$.

It can be proved (Vyridís, 2002) that the expression

$$\|\vec{u}\| = \left[\int_\Omega \alpha_{ijkl}(x)\xi_{ij}(\vec{u})\xi_{kl}(\vec{u})dx + \int_\Gamma \delta_i\delta_i\vec{u}\delta_j\delta_j\vec{u}ds\right]^{1/2}$$

defines a norm in the Hilbert space H and thus the functional F can be written as

$$F[\vec{u}] = \|\vec{u}\|^2 = (\vec{u},\vec{u})_H,$$

and trivially satisfies the Skrypnik's conditions (2.3).

The functional G is differentiable due to the smoothness of function q. The fact that the functional G is weakly continuous, and its differential is locally Lipschitz continuous, comes from the Sobolev embedding theorem of the space $W_2^2(\Gamma,\mathbb{R}^2)$ into the space $C(\Gamma,\mathbb{R}^2)$. Expression (2.4) holds, where

$$\left(A\vec{u},\vec{w}\right)_H = \int_\Gamma q_{u^iu^j}(\vec{0},x)u^iw^j\,ds,$$

and

$$\left(N(\vec{u}),\vec{w}\right)_H = \int_\Gamma \left[q_{u^i}(\vec{u},x) - q_{u^iu^j}(\vec{0},x)u^j\right]w^i ds,$$

for all $\vec{w}\in H$. The operator $A: H\to H$ is linear and symmetric. The same embedding theorem implies that the operator A is also compact. Finally, the estimate (2.5) holds due to the above embedding theorem for $p=2$.

By the proposition (2.1) we obtain that λ_0 is a bifurcation point of equation (3.2) if and only if there exists $\vec{u}_0\in H$ with $\vec{u}_0\neq\vec{0}$ satisfying the identidy

$$\int_\Omega \alpha_{ijkl}(x)\xi_{ij}(\vec{u}_0)\xi_{kl}(\vec{v})dx + \int_\Gamma \delta_i\delta_i\vec{u}_0\delta_j\delta_j\vec{v}ds$$

$$-\lambda_0\int_\Gamma q_{u^iu^j}(\vec{0},x)u_0^iv^j\,ds = 0, \qquad (3.3)$$

for all $\vec{v}\in H$. Equivalently we obtain:

$$\vec{u}_0 - \lambda_0 A\vec{u}_0 = 0.$$

Assuming that the coefficients a_{ijkl} and the boundary $\partial\Omega$ are sufficiently smooth, using the formula of integration by parts (Giusti, 1977).

$$\int_{\partial\Omega} u\delta_i v ds = \int_{\partial\Omega} Kuvds - \int_{\partial\Omega} v\delta_i uds, \qquad (3.4)$$

where K is the curvature of the curve $\partial\Omega$, equation (3.2) can be written in the equivalently elliptic boundary value problem

$$\frac{\partial}{\partial x_l}(\alpha_{ijkl}(x)\frac{\partial u^i}{\partial x_j}) = 0, \qquad x\in\Omega, \qquad k=1,2,$$

$$\alpha_{ijkl}(x)\frac{\partial u^i}{\partial x_j}n^l + D^2u^k - \lambda q_{u^k}(\vec{u},x) = 0, \quad x\in\Gamma, \quad k=1,2$$

$$\vec{u} = 0 \qquad x\in\Gamma_1,$$

where $D=\delta_i\delta_i$. In the same way equation (3.3) is equivalent to the boundary value problem

$$\frac{\partial}{\partial x_l}(\alpha_{ijkl}(x)\frac{\partial u_0^i}{\partial x_j}) = 0, \qquad x\in\Omega, \qquad k=1,2$$

$$\alpha_{ijkl}(x)\frac{\partial u_0^i}{\partial x_j}n^l + D^2u_0^k - \lambda_0 q_{u^ku^j}(\vec{0},x)u_0^j = 0,$$

$$x\in\Gamma,\ k=1,2$$

$$\vec{u}_0 = 0 \qquad x\in\Gamma_1,$$

Consider the equation (3.2) under the existence of the constraint with the property of leaving the area of the domain Ω invariant. The map

$$y:\partial\Omega\to\mathbb{R}^2 \qquad x\mapsto y(x) = x+\vec{u}(x),$$

where $\vec{u}\in H$, leaves the area of the domain Ω invariant if

$$\int_\Omega dx = \int_\Omega det\frac{\partial(y^1,y^2)}{\partial(x^1,x^2)}dx.$$

This condition is valid if

$$\Phi[\vec{u}] = 0, \qquad (3.5)$$

where

$$\Phi[\vec{u}] = \int_\Omega [div\vec{u} + \frac{\partial}{\partial x_1}(u^1u^2_{x_2}) - \frac{\partial}{\partial x_2}(u^1u^2_{x_1})]dx$$

$$= \int_{\partial\Omega}(\vec{u}\vec{n} + u^1u^2_{x_2}n^1 - u^1u^2_{x_1}n^2)ds,$$

or equivalently

$$\Phi[\vec{u}] = \int_{\partial\Omega}[\vec{u}\vec{n} + u^1(n^1\delta_2u^2 - n^2\delta_1u^2)]ds. \qquad (3.6)$$

The functional $\Phi: W_2^2(\partial\Omega,\mathbb{R})\to\mathbb{R}$ is differentiable. Using the analysis of $\vec{u}$ on the boundary $\partial\Omega$

$$\vec{u}(x) = \varphi(x)\vec{\tau}(x) + \psi(x)\vec{n}(x), \qquad \varphi,\psi\in W_2^2(\partial\Omega), \qquad x\in\partial\Omega,$$

where $\vec{\tau}(x)$ is the unitary tangential vector to Γ, we obtain

$$\Phi'[\vec{0}]u = \int_{\partial\Omega} \psi ds,$$

$$H_1 = Ker\Phi'[\vec{0}] = \{u \in H, \quad \int_{\partial\Omega} \psi ds = 0\},$$

$$H_2 = H_1^{\perp} = \{u \in H, \quad \vec{u}(x) = \varphi(x)\vec{\tau}(x) + C\vec{n}(x), \quad x \in \partial\Omega\},$$

where C is a constant. According to the Lyapunov-Schmidt decomposition, the solutions of equation (3.5) can be expressed by

$$\vec{u} = \vec{v} + h(\vec{v}), \quad v \in H_1 \text{ with } h(\vec{0}) = 0, \quad h'(\vec{0}) = 0.$$

In this case the critical points in the space H_1 of the energy functional $I[\lambda, \vec{u}]$ satisfy the integral equation

$$\int_{\Omega} a_{ijkl}(x)\xi_{ij}\left(\vec{w} + h'(\vec{v})\vec{w}\right)\xi_{kl}\left(\vec{v} + h(\vec{v})\right)dx$$

$$+\int_{\Gamma} \delta_i\delta_i\left(\vec{v} + h(\vec{v})\right)\delta_j\delta_j\left(\vec{w} + h'(\vec{v})\vec{w}\right)ds$$

$$-\lambda\int_{\Gamma} q_{u^i}\left(\vec{v} + h(\vec{v}), x\right)(w^i + (h'(\vec{v})\vec{w})^i)ds = 0\,, \qquad (3.8)$$

for all $\vec{w} \in H_1$.

Applying the proposition (2.2) we conclude that λ_0 is a bifurcation point of the equation (3.8) if and only if there exists $\vec{v}_0 \in H_1$ with $\vec{v}_0 \neq \vec{0}$ satisfying the equation:

$$\int_{\Omega} \alpha_{ijkl}(x)\xi_{ij}(\vec{v}_0)\xi_{kl}(\vec{w})dx + \int_{\Gamma} \delta_i\delta_i\vec{v}_0\,\delta_j\delta_j\vec{w}ds$$

$$-\lambda_0\int_{\Gamma} q_{u^iu^j}(\vec{0}, x)v_0^i w^j\, ds = 0\,, \qquad (3.9)$$

for all $\vec{w} \in H_1$.

The integral equation (3.9) is equivalent to the boundary value problem

$$\frac{\partial}{\partial x_l}(\alpha_{ijkl}(x)\frac{\partial v_0^i}{\partial x_j}) = 0, \quad x \in \Omega$$

$$\alpha_{ijkl}(x)\frac{\partial v_0^i}{\partial x_j}\tau^k n^l + 2\tau^k D^2 v_0^k - \lambda_0 q_{u^iu^k}(0, x)\tau^k v_0^i = 0,$$

$$\alpha_{ijkl}(x)\frac{\partial v_0^i}{\partial x_j} n^k n^l + 2n^k D^2 v_0^k - \lambda_0 q_{u^iu^k}(0, x)n^k v_0^i = C,$$

$$v_0 = 0, \quad x \in \Gamma_1.$$

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