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"Alternative Formulations of Gravity"

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In this work will review some of the different alternative formulations of gravitation. Formulations inspired by the MacDowell-Mansouri formulation of gravity, which make an attempt to express General Relativity as a gauge theory will be revised. In addition, the conditions in the symmetry breaking in the MacDowell-Mansouri action will be analyzed. This symmetry breaking allows to obtain the action of General Relativity from the MacDowell-Mansouri action, modulo additional terms, known as topological invariants, which do not affect the equations of motion of the theory.

On the other hand, alternative approaches to gravity which consider the gravitational interaction as an emerging phenomenon of thermodynamics have become relevant. From the point of view of these approaches, gravity, being an emergent interaction, lacks a quantum description, which would explain the well-known problem of the adequacy of a quantum theory of gravitation. Using the ideas of emergent gravity, through a modified entropy-area relationship, we will present a model that adds corrections to the Newtonian gravitational force. These corrections can be used to explain the anomaly of the rotation curves of galaxies, a phenomenon attributed mainly to dark matter. Finally, taking these same ideas to the cosmological scenario, we will find a modified Friedmann equation, whose solution exhibits a behavior in accordance with the usual Friedmann-Robertson-Walker cosmological model for small times, while, for large times, this solution behaves like a universe dominated by dark energy. With this, we will find an effective cosmological constant in terms of the free parameter of the model. From this perspective, a plausible entropic origin of the cosmological constant is proposed.

Dedicada a mi madre Tere y mi hermano Iñaqui...

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Chapter 1

Introduction

The logical structure of General Relativity (GR) is one of the greatest achievements in theoretical physics and is used to describe many phenomena related to the gravitational interaction, and the current results from gravitational wave astronomy cements GR as the appropriate theory to describe the gravitational interaction, together with the former classical test of GR [1], and the prediction of black holes, the theory has been proven to be the best and more accurate description of gravity at the classical level. Even if the open problem of dark energy and dark matter can be made compatible with GR (if one proposes new exotic sources of matter and energy), observations do not discard alternative theories of gravity. Although the current detections by LIGO impose strict constraints on alternative theories of gravity, further developments on gravitational wave astronomy are needed to establish GR or an extension of GR as the correct formulation for the gravitational interaction [2].

Despite the multiple accomplishments of the theory, there are some aspects of the nature of gravity that the GR cannot explain. For example, the problem of the absence of a theory describing gravity at a quantum mechanical level, that is, at a fundamental level, just as theories like quantum electrodynamics (QED) and quantum chromodynamics (QCD) do for other fundamental interactions of nature, is well known. Having a theory that describes gravity at both, classical and quantum level is fundamental, in order to fully understand the physics of objects such as black holes, or events where GR fails to explain, such as the big bang. Due to this problem, relevant theories have been developed over the years that try to offer a quantum description of gravity, the most remarkable being String Theory or Loop Quantum Gravity. These theories have offered important theoretical advances, however, the absence of experimental evidence to support their results is well known, for which there is controversy regarding these theories. Likewise, in order to understand gravity at a fundamental level, and due to important advances in differential geometry, topology, and analysis, the community has been motivated to study the gravitational interaction from another perspective perhaps different from the originally conceived by Einstein, this approaches attempt grasp the gravitational interaction at a deeper and more general geometric level. For this reason, several alternative formulations to GR have been carried out continuously. These formulations involve deeper mathematical aspects of the theory and, moreover, they could reveal a structure that helps to rethink the problem of the quantization of gravity.

It took decades until physicists understood that all known fundamental interactions can be described in terms of what we know as gauge theories. Actually, The gauge theories that emerged in a slow and complicated process gradually from GR, and the common geometrical structure between these theories. It took several decades until the importance of the gauge symmetry principle, in its generalized form to non-Abelian gauge groups developed by Yang, Mills, and others, became also fruitful for a description of the weak and strong interactions. What is currently known as Yang–Mills theories are gauge theories which describe the behavior of elementary particles using non-abelian Lie groups and they are are the mainstay for the unification of the electromagnetic and weak interactions

In 1977, S. W. MacDowell and F. Mansouri developed a novel work[3] currently known as the MacDowell-Mansouri formulation (MM), in which, for the first time, an attempt is made to conceive GR as a gauge theory, in other words, through a deeper and more detailed geometric formulation, it is intended to write the GR action as a Yang-Mills action. The idea behind this attempt is simple to understand: we know that the theories describing the other fundamental interactions of nature are gauge theories, and we know how to quantize those theories. Therefore, if one manages to express gravity as a gauge theory, then, in principle, its quantization would be achievable. However, this method is not easy to perform, the reality is that, due to the aspects that make the gravitational interaction different from the rest of the interactions, and other difficulties that arise, such as the well-known non-renormalization of gravity[4], it has not been possible to carry out this idea, however these developments have allowed us to know more about the structure of gravitational interaction.

In the MacDowell-Mansouri formulation, the action of GR emerges from a more general action, which is invariant under the $SO(4,1)$ group, called the Anti-de Sitter (AdS) group. (it can also be considered the group $SO(3,2)$) When the $SO(4,1)$ symmetry is broken down to the $SO(3,1)$ group, called the Lorentz group, the MM action splits into the action of GR with cosmological constant and a topological term, the so-called Euler's term, or Gauss-Bonett term. Topological terms do not affect the equations of motion of the classical theory, however, these terms have been shown to have relevance at the semi-classical level. For example, the so-called θ-ambiguity is well known in QCD [5], this ambiguity originates from the Pontryagin topological term. In gravity, the Pontryagin term is related to the well-known problem of the Immirzi parameter [6, 7]. The Immirzi parameter is highly relevant in theories such as Loop Quantum Gravity. Viewed in this way, a gauge invariant action for gravity is achieved, being the Lorentz group the gauge group. Thus, instead of modeling GR by means of an action whose fundamental field is the metric, it is modeled by means of an action whose fundamental fields are the tetrad field and the spin connection field.

Likewise, the MM formulation has been the inspiration for the development of other Gauge formulations of gravity, highlighting, for example, the one carried out in [8], in which, an MM-type action featuring self-dual forms is built. This formulation naturally includes the Pontryagin topological term in addition to the Euler term, featuring the fact that the action is quadratic in the strength field, thus, having a Yang-Mills action-type structure. Moreover, as pointed out by the authors, this formulation may contribute to understanding the fact that the physical states of quantum general relativity are the exponential of the so-called Chern-Simons action.

As mentioned before, in order to obtain GR from the MM action, it is necessary to break the $SO(4,1)$ symmetry down to $SO(3,1)$, however, this breaking must be done "by hand". That is, there is no known process that allows to naturally break down the symmetry (for example, as in the case of the spontaneous gauge symmetry breaking in the electro-weak interaction). This is why there have been several attempts with the purpose of equipping the formalism with something that allows the breaking of the symmetry in a less artificial way. A remarkable attempt was made by K.S. Stelle and P. C. West [9]. This formulation adds in the action an $SO(4,1)$ vector, and the idea is that, when setting a privileged direction for this vector, then the breaking of the symmetry occurs. Inspired by these ideas, in [10] it is analyzed in more depth what dynamic conditions must be met by this new geometric element, which contains the relevant geometric information, that is, for example, the space-time metric can be expressed in terms of this object.

The alternative formulations of GR, such as the MacDowell-Mansouri formulation, among others have worked out to reveal in a clearer way the structure, both physical and mathematical, of the gravitational interaction, at the same time, they give us clues about a possible way forward to achieve a theory that describes gravity at a quantum level, however, it is clear that this task has not yet been achieved. On the other hand, there are entities such as dark matter, which cannot be integrated and described in these alternative theories of gravitation (Even trying to include ordinary matter, for example fermions, is a complicated task to do). Also, although expressions for the cosmological constant can be obtained in these formulations, it is impossible to detail the dynamics when considering dark energy. For such reasons, there is the need to consider other alternatives, more concretely phenomenological models that, in some way, do include the dark sector and, tell us something about the quantization of gravity, or even more, we can ask ourselves: Is gravity really quantizable? This is a controversial question, and, if the answer were negative, this would imply that gravity is not a fundamental interaction, but rather is some effective interaction. In this way, any attempt to obtain a quantum theory of gravity would be sadly fruitless. As evidence of these ideas, in 1995, in a masterly work [11], T. Jacobson showed that the GR field equations can be obtained from the fundamental thermodynamic relation $TdS = \delta Q$, also called the Clausius relation. This result is important since it constitutes a first indication that gravity is not a fundamental interaction, but an effective interaction arising from thermodynamics. As mentioned in this paper: "in this sense, quantizing gravity is analogous to the case of sound in a gas propagating as an adiabatic compression wave: Since the sound field is only a statistically defined observable, it should be no canonically quantized as if it were a fundamental field, even though there is no doubt that the individual molecules are quantum mechanical. This suggests that it may not be correct to canonically quantize the Einstein equations, even if they describe a phenomenon that is ultimately quantum mechanical".

In the spirit of Jacobson's work, recently, in a remarkable work [12], E. Verlinde, starting from first principles and general assumptions, shows that Newton's law of gravitation arises naturally and unavoidably in a theory in which space is emergent through a holographic scenario. In Verlide's scheme, gravity is explained as an entropic force caused by changes in the information associated with the positions of material bodies. (and, a relativistic generalization of these ideas directly leads to the Einstein equations) In this sense, the scheme presented by Verlinde offers a powerful tool to study gravity modifications from a thermodynamic perspective. For example, given that in this approach, the dynamics are encoded in the area-entropy relationship, it is possible to obtain modifications to Newton's Law of Gravitation from modifications to the area law for Bekenstein's entropy- Hawking [13]. These modifications can be related to theories that try to alternatively explain phenomena associated with dark matter, such as the discrepancy between the predicted and actual rotation curves of galaxies. [14, 15]

On the other hand, the several proposals to solve what is currently known as the dark energy problem can be divided in two classes. In the first class we remain in the context of general relativity (GR) and assume that this acceleration is produced by an unknown energy density or by the existence of a new type of matter that has the nonphysical property of negative pressure. In this context, it seems that the best candidate is the cosmological constant Λ. Even if the observations are compatible with the existence of Λ there are serious theoretical problems to consider this model as the final answer [16, 17]. The second class, assumes that we do not need a new energy density with strange physical properties, but instead considers that this acceleration is a consequence of a modified theory of gravity. For the last decade, there has been a lot of work in modified theories of gravity [18], constructing models that exhibit an accelerating scale factor in their late time dynamics [19]. The most successful attempts are Horndenski type theories [20]. In summary one can address the dark energy problem by either staying in the framework of GR and introducing a dark energy component, or be confident with our knowledge of the energy and matter content of the universe and modify the gravitational interaction. If we entertain the possibility that the dark energy problem is a consequence of the poor understanding of gravity, we need to formulate an alternative to GR. In a more recent paper [21], the author explores the possibility of a common origin of dark matter and dark energy in the context of an emergent formulation of gravity. The dark matter predictions for this theory have been put to test in several works [22, 23]. However, even if the specific formulation of Verlinde is not the last answer, the entropic or emergent origin of gravity and its connection to the dark matter and dark energy can shed some light to origin of the dark sector of the Universe.

The ideas of entropic gravity have been extended to cosmology. In recent works [24–27], it has been possible to obtain the well-known Friedmann equations from thermodynamic considerations, in the spirit of Jacobson's ideas, one can obtain the Friedmann equations of a Friedmann-Robertson-Walker (FRW) universe by applying the Clausius relation to the apparent horizon of the FRW solution. The entropy of the horizon is assumed to be proportional to its area. Under the same argument as in the case of Verlinde's emergent gravity, if one considers deviations from the area law for the entropy, then one should expect to obtain a modified Friedmann equation whose modifications are traced to the modifications of the entropy-area relationship. In a recent work[28] it has been shown that, for an entropy-area relationship that contains a term proportional to the volume of the horizon, the solution to the resulting modified Friedmann equation exhibits an exponential behavior for large times, that is, this solution mimics the behavior attributed to dark energy. From this point of view it can be argued that what is currently known as dark energy may have an entropic origin.

This thesis is structured as follows: In Chapter 2 we present a review of the basic concepts and ideas of gravitation, first motivating the development of GR from Newtonian gravitation and finally, giving a brief study of the main ideas concerning GR.

In chapter 3, some formulations of gravity such as MM, self-dual MM, and Stelle-West-inspired formulations will be reviewed.

In Chapter 4 we will present our work about the symmetry breaking conditions in MacDowell-Mansouri-type actions. This is part of the work done in the doctoral process.

In chapter 5 the thermodynamic formulation of gravity is addressed. Jacobson's ideas will be presented, as well as Verlinde's ideas about entropic gravity.

Chapter 6 corresponds to the other part of the original contribution developed in the doctoral process, in this chapter, firstly, we obtain modifications to the Newtonian gravitational force will be obtained from modifications to the area law for the area-entropy relationship. Lastly, by taking the ideas of entropic gravity to the cosmological scenario, a modified Friedmann equation will be obtained and solutions to that equation will be presented.

Finally, chapter 7 is devoted to conclusions and final remarks.

Chapter 2

A Review on Classical Gravitation

"Gravitation surely is not the most appreciated of all forces. In its absence, my first steps as a child would have been far easier to achieve, and my clumsiness would have less practical consequences. This is unfair judgment though, for without gravitation there would be no one to enjoy floating around. Without gravitation, the splendid structures of our Universe could not have formed. Without gravitation, galaxies would not swirl and stars would not shine; planets would never have come to existence, and life would not be. Because of its evidence and ubiquity in our daily experience of motion, it is not surprising that gravitation was the first physical interaction ever described within a solid scientific framework. Newton's theory of the universal attraction of massive bodies was, at the end of the 17th century, a proper scientific revolution. It remains one of the best examples of conceptual unification – how audacious was it to claim that objects falling on the ground and the orbits of celestial bodies are merely two facets of the same phenomenon? Albeit unchallenged for more than two centuries, Newton's formulation of physics was only a prelude. In the early 20th century, another revolution occurred, and dramatically changed our conception of the Universe. With the advent of Einstein's relativity, the hitherto distinct concepts of space and time merged into a hybrid structure called spacetime. Furthermore, this space-time turned out to be somehow malleable, gravity being nothing but its geometry. This superb theory, formulated in 1915, was not less superbly confirmed, in 1919, by Eddington's measurement of the deflection of starlight by the Sun. Besides light bending, relativity also predicted some exotic phenomena, among which gravitational waves and black holes. The first ones, which are to gravity what light is to electromagnetism, were first detected in 2015, that is, exactly one century after the formulation of the theory encompassing them. As these gravitational waves were produced by the collision of two black holes, they also provided indirect proof of their existence; and if this does not convince you, take a look at the 2019 photograph of the M87* supermassive black hole! That picture, whose interest relies on the deviation of light by the black hole, remarkably marked the centenary of Eddington's observation. Could there be a better occasion to start a journey around the world of gravity? Be careful though, as you may fall in love with it, just like I did" . . . Pierre Fleury.

2.1 Newtonian Gravity

In his celebrated work Philosophiæ Naturalis Principia Mathematica, or "Principia", Isaac Newton established that material bodies interact gravitationally exerting a force of magnitude given by

$$F = \frac{Gm_1m_2}{r^2} \tag{2.1}$$

where G is the gravitational constant $G \approx 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$. Taking this into account, Newton was able to give a theoretical agreement with the previous results empirically found by Kepler in 1608.

It is simply to write down Newtonian gravity as a field theory. Let ρ describe the matter distribution, and consider a test particle of mass m (the mass m is often referred as "mass charge"). Let this particle be located at some point represented by the vector $\vec{x}$. Then, the force acting on the particle is given by

$$\vec{F} = -m\nabla\Phi, \tag{2.2}$$

where Φ is called the gravitational field. Moreover, we can write

$$\nabla^2\Phi = 4\pi G\rho, \tag{2.3}$$

which is the gravitational Poisson equation. For a sphere centered at a particle of mass M, the Poisson equation (2.3) gives

$$\Phi = -G\frac{M}{r}, \tag{2.4}$$

which corresponds to the gravitational potential of a point mass.

Then, according to Newton, the equation of motion of the particle is given by

$$m_i\ddot{\vec{x}} = -m_g\nabla\Phi \tag{2.5}$$

where, now m_i stands for the inertial mass and m_g stands for the gravitational mass or the "gravitational charge". Surprisingly, there exists an equivalence between these two masses. It turns out that

$$m_i = m_g \tag{2.6}$$

The observation that all particles fall the same way has led to the formulation of so called equivalence principles. It is common to distinguish between three versions which we can summarize as follows

1. Weak Equivalence Principle: "Freely falling bodies with negligible gravitational self interaction follow the same path if they have the same initial velocity and position".

2. Einstein equivalence principle: "In a local inertial frame, the results of all non-gravitational experiments are totally indistinguishable from those of the same experiment performed in an inertial frame in Minkowski spacetime".

3. Strong equivalence principle: The gravitational motion of a small test body depends only on its initial velocity and position but not on the material constitution of the body.

For almost 300 years, Newtonian laws of motion and gravitation remained unchanged and they conformed the ultimate theory to describe the gravitational interactions of objects on earth, planets, and galaxies. However, it was a matter of time before some problems with Newtonian gravity arose, and therefore, it was discovered that there was a need for a more complete theory describing the gravitational interaction. Firstly, the lack of Lorentz invariance in Newtonian mechanics offers a clear incompatibility between the latest and Special Relativity. Another failure of Newtonian mechanics appears when trying to explain the precession of the orbit of mercury, concretely, there is a quantity of 43 arcsec/century in the precession of the orbit that remain unexplained by using Newtonian mechanics. Therefore, this was an indication that another theory of gravity was needed. With the further development of GR, this phenomenon could be explained.

2.2 General Relativity

The theory of GR is a classical theory of gravity and so far the most successful for describing gravitational interaction. This theory is based primarily on the equivalence principle, which states that a system immersed in a gravitational field is punctually indistinguishable from an accelerated non-inertial reference system, that is, gravitational forces and inertial forces are equivalent. GR relates gravity to purely geometric issues of a 4-dimensional space called spacetime, that is, gravity is generated by the curvature of that spacetime which in turn is due to the presence of matter, in other words, matter curves into spacetime, causing the presence of gravity. Mathematically speaking, spacetime is regarded as a differentiable manifold, but for the moment, we will not focus on such deep mathematical concepts, for the moment it will be enough to define the mathematical objects needed to describe the theory.

In GR, the dynamics of spacetime is encoded in a second-rank tensor called the metric tensor $g_{\mu\nu}$, being, in this formulation, the fundamental object in which, all information of spacetime is stored. Metric tensor allows to define the so called line element ds, interpreted as the infinitesimal distance between to points in the manifold. For an n-dimensional space with coordinates x^α the linea element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0.1, ..., n, \tag{2.7}$$

GR is governed by Einstein's field equations, which relate the curvature of spacetime with the content of matter, since, as mentioned, in this theory the gravitational field is the effect of spacetime curvature due to the presence of matter. In the classical non-relativistic limit, that is, at small speeds compared to the speed of light and weak gravitational fields, Einstein's field equations are reduced to the Poisson equation for the gravitational field that is equivalent to Newton's law of gravitation. As mentioned before, GR is strongly based on the equivalence principle, which is the statement that the laws of physics, in small enough regions of spacetime, look like those in special relativity (SR). Interpreting this in a mathematical fashion, regarding spacetime as a (pseudo) Riemannian manifold, this tell us that these laws, when written in normal Riemannian coordinates x^μ at some pint p, are described by equations which take the same form as they would have in flat space (later this will be better understood in terms of tangent spaces). The simplest example is that of freely-falling particles (this is why sometimes these are called freely-falling coordinates). In flat space such particles follow straight-lines. If the curve is parametrized as $x^\mu = x^\mu(\lambda)$, this statement means that the second derivative of the parametrized line vanishes

$$\frac{d^2x^\mu}{d\lambda^2} = 0, \tag{2.8}$$

and according to the equivalence principle, this equation should hold in curved space as long as the coordinates are normal Riemannian coordinates. For general coordinates, there would be a more general equation which should reduce to equation (2.8) for flat space. In fact, one can find such equation, by minimizing the distance between to points in an arbitrary space, of course, for this distance we use the line element, equation (2.7), this means, given the distance

$$I[x^\mu, \dot{x}^\mu] = \int \sqrt{g_{\mu\nu}(x^\alpha)\dot{x}^\mu\dot{x}^\nu}d\lambda, \tag{2.9}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, we can use standard techniques of variational calculus in order to obtain the extremal and obtain the equation giving the path corresponding to minimal length between the two points. In equation (2.9) we identify the Lagrangian $\mathcal{L} = \sqrt{g_{\mu\nu}(x^\alpha)\dot{x}^\mu\dot{x}^\nu}$, although, for simplicity, it is customary to use not this lagrangian, but $\mathcal{L} = \frac{1}{2}g_{\mu\nu}(x^\alpha)\dot{x}^\mu\dot{x}^\nu$. Hence, using the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = 0 \tag{2.10}$$

one obtains the well-known geodesic equation

$$\frac{d^2x^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{2.11}$$

A geodesic is a curve representing in some sense the shortest path between two points in a Riemannian manifold. It is a generalization of the notion of a straight line to a more general setting. Therefore, in GR, free particles move along geodesics. In equation (2.11) the quantities $\Gamma^\sigma_{\mu\nu}$ are to the metric tensor via

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\kappa}\left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right). \tag{2.12}$$

This is known by different names: sometimes the Christoffel connection, sometimes the Levi-Civita connection, sometimes the Riemannian connection, and this is one of the most important formulas in this subject, since this connection is the one on which conventional general relativity is based. Connections are important objects in differential geometry, the study of manifolds with metrics and their associated connections is called "Riemannian geometry". So far, we have argued that curvature of spacetime is a necessary ingredient to describe gravity, and, to show that this is a sufficient condition, we can analyze how the results of Newtonian gravity fit into the picture. In order to define the Newtonian limit, we consider the following requirements: The speed of the particles is small compared with the speed of light, secondly, we require the gravitational field to be weak, this means, we regard it as a small perturbation of flat space. Finally we consider this field to be static, that is, independent of time. We impose this conditions in the geodesic equation (2.11), where we take the parameter λ to be the proper time τ. The first statement means that

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau}, \tag{2.13}$$

so the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{2.14}$$

Now, since we do not have explicit dependence of time, the relevant Christoffel symbols, from (2.12) simplify to

$$\begin{aligned}\Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\lambda}\left(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right)\\ &= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}.\end{aligned} \tag{2.15}$$

Finally, the weakness of the gravitational field can be expressed as a decomposition of the metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.16}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (flat metric), and $h_{\mu\nu}$ is a small perturbation $|h_{\mu\nu}| << 1$. From the condition $g_{\mu\nu}g^{\mu\alpha} = \delta^\alpha_\nu$ we find $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Indices of the perturbation metric can be raised or lowered using the flat metric. Using the decomposition (2.16), we obtain

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}, \tag{2.17}$$

and the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2, \tag{2.18}$$

using $\partial_0 h_{00} = 0$, the $\mu = 0$ component of this equation is

$$\frac{d^2t}{d\tau^2} = 0, \tag{2.19}$$

which has the solution $\frac{dt}{d\tau} = constant$. As for the spatial components of equation (2.18), we take into account that the spatial components of $\eta^{\mu\nu}$ are just those of the 3×3 identity matrix, therefore, this equation becomes

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2\partial_i h_{00}, \tag{2.20}$$

which, after dividing by $\left(\frac{dt}{d\tau}\right)^2$ can be written as

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}, \tag{2.21}$$

where, for the derivative of the left hand side, we have used to chain rule to convert it into a derivative with respect to t. So far we have achieved an important result. Note that the left hand side of the latest equation is just an acceleration, hence, comparing with the Newtonian equation

$$a^i = -\partial_i \Phi, \tag{2.22}$$

where Φ is a scalar field regarded as the gravitational field, we identify

$$h_{00} = -2\Phi, \tag{2.23}$$

or

$$h_{00} = -(1 + 2\Phi). \tag{2.24}$$

Therefore, this interpretation of gravity as the curvature of space time is indeed sufficient to describe gravity in the Newtonian limit. What comes next is the employment of the Principle of Covariance, this principle can be seen as the Einstein equivalence principle plus the intuitive requirement that the laws of physics be independent of the used coordinates. Using this argument we can think of this as rule to take equations in flat space into the form they would have in general curved space. For instance, it is well known that ordinary partial derivative ∂_μ should be changed to covariant derivative ∇_μ, where the covariant derivative of a tensor of rank $r+s$ is

$$\begin{aligned}\nabla_\beta A^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} =& \partial_\beta A^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} + \Gamma^{\mu_1}_{\beta\gamma} A^{\gamma \ldots \mu_r}_{\nu_1 \ldots \nu_s} + \ldots + \Gamma^{\mu_r}_{\beta\kappa} A^{\mu_1 \ldots \mu_{r-1}\kappa}_{\nu_1 \ldots \nu_s} \\ &- \Gamma^{\rho}_{\beta\nu_1} A^{\mu_1 \ldots \mu_r}_{\rho \ldots \nu_s} - \ldots - \Gamma^{\lambda}_{\beta\nu_s} A^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_{s-1}\lambda}.\end{aligned} \tag{2.25}$$

For instance, if we consider the conservation of energy in flat space, written as $\partial_\mu T^\mu = 0$, the corresponding law in curved space is just

$$\nabla_\mu T^\mu = 0. \tag{2.26}$$

Of course, one has to be careful with this rule, since it is not a universal rule, that is, there might be some ambiguities when translating equations from flat space into equations in curved space. For example, consider some law written in flat space as $Y^\mu \partial_\mu \partial_\nu X^\nu = 0$. We know that partial derivatives commute, however, covariant derivatives do not. If one simply replaces these partial derivatives for covariant derivatives, we would have a problem of ordering, The prescription for generalizing laws from flat to curved spacetimes does not guide us in choosing the order of the derivatives, There exist several techniques for dealing with such ambiguities, however, there is no way to resolve these problems by pure thought alone; the fact is that there may be more than one way to adapt a law of physics to curved space, and ultimately only experiment can decide between the alternatives. Actually, one can find that the commutator between covariant derivatives yields

$$[\nabla_\mu, \nabla_\nu]\, \omega^\rho = R^\rho_{\mu\nu\alpha} \omega^\alpha, \tag{2.27}$$

where $R^{\rho}_{\mu\nu\alpha}$ is the Riemann curvature tensor, given by

$$R^{\sigma}_{\mu\gamma\nu} \equiv \partial_{\gamma}\Gamma^{\sigma}_{\mu\nu} - \partial_{\nu}\Gamma^{\sigma}_{\mu\gamma} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\gamma} - \Gamma^{\kappa}_{\mu\gamma}\Gamma^{\sigma}_{\kappa\nu}, \tag{2.28}$$

A flat space is one for which all the components of the Riemann tensor vanish, while for a curved space not all of its components are zero, in other words, having a non-zero Riemann tensor is a necessary and sufficient condition for having a curved space. The Riemann tensor in n dimensions has n^4 components, however, its symmetry properties imply that in the end it has only $n^2(n^2-1)/12$ independent components. Therefore, in 4 dimensions the Riemann tensor has 256 components of which only 20 are independent. The Riemann tensor satisfies the symmetry property

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\nu\mu\alpha\beta}, \tag{2.29}$$

where $R_{\alpha\beta\mu\nu} = g_{\alpha\sigma}R^{\sigma}_{\beta\mu\nu}$. In addition, it obeys a cyclic relationship as follows

$$R^{\alpha}_{\beta\mu\nu} + R^{\alpha}_{\mu\nu\beta} + R^{\alpha}_{\nu\beta\mu} = 0. \tag{2.30}$$

Another important property of the Riemann tensor is that it satisfies

$$\nabla_{\lambda}R_{\rho\sigma\mu\nu} + \nabla_{\rho}R_{\sigma\lambda\mu\nu} + \nabla_{\sigma}R_{\lambda\rho\mu\nu} = 0, \tag{2.31}$$

and this is known as the Bianchi identity. This identity is intimately related to the fact that the commutator of covariant derivatives satisfies the Jacobi identity

$$[[\nabla_{\lambda}, \nabla_{\rho}], \nabla_{\sigma}] + [[\nabla_{\rho}, \nabla_{\sigma}], \nabla_{\lambda}] + [[\nabla_{\sigma}, \nabla_{\lambda}], \nabla_{\rho}] = 0. \tag{2.32}$$

Having established how physical laws govern the behavior of fields and objects in a curved spacetime, we can complete the establishment of GR by introducing Einstein's field equations, which dictates the dynamics of spacetime, via how the metric responds to energy and momentum present in spacetime. The way Einstein originally realized the problem was that one would like to obtain an equation that replaces the Poisson equation for the Newtonian gravitational field

$$\nabla^2\Phi = 4\pi G\rho. \tag{2.33}$$

Thinking of this as a limit equation, using the previous results and regarding the mass density ρ and as the T_{00} component of a tensor $T_{\mu\nu}$, we see that the general equation we should have should predict the equation

$$\nabla^2 h_{00} = -8\pi G T_{00}, \tag{2.34}$$

of course, in order to generalize this equation to curved space, we could think of replacing the second order operator in the left hand side by covariant derivatives, however, it turns that $\nabla g = 0$ (this is also known as metric compatibility). The other approach is to consider a quantity involving derivatives of second order of the metric. We see, from equations (2.28) and (2.12) that the Riemann tensor, indeed has the required derivatives, however, the indices counting does not match. Nevertheless we can construct a "required-index" tensor obtained by a contraction of the Riemann tensor in the following way

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\beta}_{\beta\mu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\kappa}_{\mu\xi}\Gamma^{\xi}_{\kappa\nu}. \tag{2.35}$$

This is a symmetric tensor $R_{\mu\nu} = R_{\nu\mu}$, and in 4 dimensions it has 10 independent components. It is important to note that in general, having a vanishing Ricci tensor for a certain space does not mean that this space is flat, since this does not imply that the Riemann tensor is also vanishes. Hence, the candidate equation would be

$$R_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.36}$$

however, energy-momentum conservation would imply $\nabla_\mu R^{\mu}_{\nu} = 0$, which, in general does not hold, since, one finds

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2}\nabla_\nu R, \tag{2.37}$$

where R is the Ricci curvature scalar and is the trace of the Ricci tensor

$$R = g_{\mu\nu}R^{\mu\nu} = g^{\mu\nu}R_{\mu\nu} = R^{\mu}_{\mu}, \tag{2.38}$$

however, if we consider the tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{2.39}$$

which, from the Bianchi identity (2.31), satisfies $\nabla^{\mu}G_{\mu\nu} = 0$, we are led to propose

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.40}$$

as the field equation for the metric. It is clear that this equation satisfies all the requirements. And these are, in fact, Einstein's Field Equations for GR. These equations, and they tell us how the curvature of spacetime reacts to the presence of energy-momentum. Einstein's Field Equations are six independent second order differential equations for the metric tensor. In vacuum, Einstein's Field Equations reduce to

$$R_{\mu\nu} = 0. \tag{2.41}$$

In general, these equations are very difficult to solve. Even the vacuum equations turn to be almost impossible to solve unless one assumes a sort of simplifying assumption, such that the metric has a significant degree of symmetry.

It is important to recall that Einstein equations can be obtained in a more modern fashion, starting from a variational principle[1]. Hilbert himself noticed that the only scalar constructed from the metric which is no higher than second order derivatives for the metric, as required, is the Ricci scalar, therefore, he proposed as the simplest choice for a Lagrangian density for GR

$$\mathcal{L}_H = \sqrt{-g}R, \tag{2.42}$$

[1] In fact the equations were first derived by Hilbert, not Einstein, and Hilbert did it using the action principle. But he had been inspired by Einstein's previous work, and Einstein himself derived the equations independently, so they are rightly named after Einstein. The action, however, is rightly called the Hilbert action

hence, the action principle for GR is

$$S_{EH}[g_{\mu\nu}] = \int_{\mathcal{M}} \sqrt{-g} R d^4x, \tag{2.43}$$

which is commonly known as the Einstein-Hilbert action, while, the Lagrangian density (2.42) is known as the Einstein-Hilbert term. The equations of motion are obtained by varying the action (2.43) with respect to the metric. This variation yields

$$\delta S_{EH} = \int_{\mathcal{M}} d^4x \left[\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta\sqrt{-g}\right] = \delta S_1 + \delta S_2 + \delta S_3. \tag{2.44}$$

after some tedious calculations one finds

$$\delta S_1 = \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_\sigma \left[g^{\mu\sigma} \left(\delta\Gamma^\lambda_{\lambda\mu}\right) - g^{\mu\nu} \left(\delta\Gamma^\sigma_{\mu\nu}\right)\right], \tag{2.45}$$

this is an integral of a total derivative, over all space, which, using the Stokes' theorem, can be converted into an integral over the boundary of $\mathcal{M}$, which can be set to zero by making the variation vanish at infinity. As for the term δS_3 , one finds

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \tag{2.46}$$

therefore the total variation of the action is

$$\delta S_{EH} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right] \delta g^{\mu\nu}, \tag{2.47}$$

and this must vanish for arbitrary variations, so we are led to Einstein's equations in vacuum

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{2.48}$$

The non-vacuum equations are obtained by adding to the Einstein-Hilbert action a term related to matter S_M. If we write the total action as

$$S = \frac{1}{8\pi G} S_{EH} + S_M, \tag{2.49}$$

the variation becomes

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \kappa \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0, \tag{2.50}$$

if we set

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{2.51}$$

Then we obtain

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^{-1} T_{\mu\nu}. \tag{2.52}$$

It is worth mentioning that it is possible to add a constant to the Einstein-Hilbert action, which conduces to interesting dynamics. If we consider the action

$$S = \int d^4x\sqrt{-g}(R - 2\Lambda) \tag{2.53}$$

where Λ is some constant. The resulting field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.54}$$

This constant Λ is the well-known and infamous cosmological constant , which was originally introduced by Einstein after it became clear that there were no solutions to his equations representing a static universe with a nonzero matter content. Currently, the cosmological constant represents one of the principal unsolved problems in physics. Although in this work we will not deal with cosmological solutions in GR, the cosmological constant will be important from a group-theoretical point of view, and, in Chapter 4 the subject will be addressed by attempting to explain the origin of the cosmological constant through entropic arguments. It is clear that obtaining the Einstein-Field equations from the Einstein-Hilbert action is, mathematically more formal, however, in this formulation, the calculations turn to be somehow tedious. In the following section we will see that, rewriting all the above in terms of the more modern and transparent language of differential geometry turns to simplify the calculations and shed a different perspective on the geometrical aspects of the theory.

Despite its many great achievements, there is evidence that suggests that GR is not the most complete theory to describe the gravitational interaction, or to give a complete explanation to certain phenomena. For example, GR has nothing to say concerning “dark matter” or “dark energy”, currently described as “mysterious” and requiring much further research. If GR is defective in the particulars cited above, how should it be reformulated? or, by what should it be replaced?

Chapter 3

Form Theories of Gravity

3.1 First Order Gravity

A common approach to GR is know as first order gravity, in which, the formulation of gravity becomes metric independent, that is, one introduces new variables, or, in other words, the geometric insight of the theory is different from the usual one. In this formulation, instead of the metric, the new relevant quantities are now the so called tetrad field and the spin connection. The orthonormal basis for the 1-forms is given by

$$\theta^a = e^a_\mu(x) dx^\mu, \tag{3.1}$$

note that Greek indices are used for spacetime objects whilst Latin indices are used for local Minkowski objects (sometimes Greek indices are referred as curved indices while Latin indices are referred as flat indices). The vielbein allows us to relate the spacetime metric $g_{\mu\nu}$ with the flat Minkowskian metric η_{ab} as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{3.2}$$

Note that, taking determinant on the last equation we get $g = -e^2$, that is, $e = \sqrt{-g}$. In this sense, the vielbeins are sometimes referred as the square root of the metric. Using the inverse e^μ_a of the vielbein, which satisfy

$$e^\mu_a e^a_\nu = \delta^\mu_\nu, \quad e^\mu_a e^b_\mu = \delta^b_a, \tag{3.3}$$

equation (3.2) can be written as

$$\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b, \tag{3.4}$$

This allows us to obtain the metric from the tetrad, and the way around. The tetrad should transform as a Lorentz vector under $SO(3,1)$ transformations, while the metric should remain invariant under such transformations, hence, we have the condition known as Lorentz condition

$$\Lambda^a_c(x) \Lambda^b_d(x) \eta_{ab} = \eta_{cd}, \tag{3.5}$$

where $\Lambda^a_b(x)$ are the corresponding Lorentz transformations. Due to the fact that this group acts separately at each point, when comparing tangents spaces at different points one needs to introduce a connection ω, and ,with this connection one can perform covariant derivatives of tensors in the tangent space similar to what one does with tensors in spacetime using the Levi-Civita connection. For example

$$\nabla_\mu X^a = \partial_\mu X^a + \omega^a_{\mu b} X^b, \quad \nabla_\mu X^a_b = \partial_\mu X^a_b + \omega^a_{\mu c} X^c_b - \omega^c_{\mu b} X^a_c, \tag{3.6}$$

and so on. Of course, ω transforms as a connection under Lorentz transformations

$$\omega^a_{\mu b} \to \Lambda^a_c \Lambda^d_b \omega^c_{\mu d} - \Lambda^c_b \partial_\mu \Lambda^a_c. \tag{3.7}$$

Using this covariant derivative, metric compatiblity $\nabla g = 0$ implies

$$\nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega^c_{\mu a} \eta_{cb} - \omega^c_{\mu b} \eta_{ac} = 0, \tag{3.8}$$

which implies

$$\omega_{\mu ab} = -\omega_{\mu ba}. \tag{3.9}$$

We can also relate the spin connection field $\omega^b_{\mu a}$ to a 1-form, to obtain what we shall refer as 1-form connection, as

$$\omega^a_b = \omega^a_{\mu b} dx^\mu, , \tag{3.10}$$

We can construct the curvature 2-form using the spin connection as [1]

$$R^{ab}(\omega) = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} := \frac{1}{2} R^{ab}_{\mu\nu}(\omega) dx^\mu \wedge dx^\nu, \tag{3.11}$$

where

$$R^{ab}_{\mu\nu}(\omega) = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^a_{\mu c} \omega^{cb}_\nu - \omega^a_{\nu c} \omega^{cb}_\mu. \tag{3.12}$$

the "usual"Riemann tensor (2.28) is obtained as

$$R^\rho_{\sigma\mu\nu}(g) = R^{ab}_{\mu\nu}(\omega(e)) e^\rho_a e_{\sigma b} \tag{3.13}$$

where $R^{ab}_{\mu\nu}$ is given by equation (3.12). Note that the form in which we construct the curvature 2-form is very similar to the way one constructs the field strength in gauge theories $F = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$, where $\mathcal{A}$ is the gauge potential, hence, in this sense, the object ω is some kind of "gravitational gauge potential'". Now we introduce another object, this is a 2-form called the torsion 2-form, defined as

$$T^a = d\theta^a + \omega^a_b \wedge \theta^b := D_\omega \theta^a, \tag{3.14}$$

where $D_\omega = d + \omega$ is the covariant exterior derivative. Equations (3.12) and (3.14) are known as the Maurer-Cartan structure equations. For the case of vanishing torsion we recover GR, and from the latest equation we can obtain a relation between the tetrad and the spin connection

$$d\theta = -\omega \wedge \theta \tag{3.15}$$

[1] We can always rise or lower latin indices using η^{ab} or η_{ab}

from this equation, when translating to components notation one obtains the well known expression for the affine connection, that is, equation (2.12), which, as we know, is the only one symmetric torsion-free, metric compatible connection. The components of the torsion are easily obtained, yielding

$$T^{\lambda}_{\mu\nu} = e^{\lambda}_{a} T^{a}_{\mu\nu} = e^{\lambda}_{a} \left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu b} e^b_\nu - \omega^a_{\nu b} e^b_\mu \right) = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}, \tag{3.16}$$

therefore, we see explicitly that for a symmetric connection, such as the Levi-Civita connection used in GR, the torsion vanishes. Although in GR one has vanishing torsion, there exist theories in which non-symmetric connections are considered, yielding to non zero torsion. even more, there are connections which yield to zero curvature, but non-zero torsion. A remarkable example is that of the Weitzenböck connection. These torsional theories conform what is known as Teleparellel Gravity [29]. There exist other formulations of gravity where an arbitrary connection is considered, then, covariant derivatives can be constructed out of this connection, as long as it is metric compatible, this is, as long as the condition $\nabla g = 0$ holds. This yields to non-symmetric part of the Ricci tensor and therefore a non-symmetric sector of the Einstein's Field equations is obtained. A simple calculation shows that

$$\begin{aligned} D_\omega R^{ab} &= 0, \\ D_\omega T^a &= R^{ab} \wedge \theta_b, \end{aligned} \tag{3.17}$$

the first equation is the generalization of equation (2.29), while the second is the Bianchi identity (2.31)[2].

With the given information, we are led to construct the so called Einstein-Cartan action, which reads

$$S = \int d^4x \epsilon^{abcd} \left(R_{\mu\nu ab}(\omega) e_{\rho k} e_{\sigma d} - \frac{\Lambda}{3} e_{\mu a} e_{\nu b} e_{\rho c} e_{\sigma d} \right) \epsilon^{\mu\nu\rho\sigma}, \tag{3.18}$$

if we rewrite this action in terms of differential forms we obtain

$$S(\omega, e) = \frac{1}{32\pi G} \int \epsilon_{abcd} \left(R^{ab}(\omega) \wedge e^c \wedge e^d - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d \right). \tag{3.19}$$

It is important to mention that in this formalism, the connection ω and the tetrad e are treated as independent variables, therefore, when varying this action we have to do it independently with respect to both quantities. Variation of this action with respect to ω gives

$$T^a = 0 \tag{3.20}$$

which is clearly recognized as the torsion free condition, which, as mentioned before, implies

$$de^a + \omega^a_b \wedge e^b = 0, \tag{3.21}$$

this establishes the relation between the tetrad and the spin connection ω

$$\omega^{ab}_{\mu}(e) = e^{\nu a} \left(\partial_\mu e^b_\nu - \Gamma^{\lambda}_{\mu\nu}(g) e^b_\lambda \right). \tag{3.22}$$

[2]Sometimes both equations are called Bianchi identities

Variation of the action with respect to θ yields

$$\left(R^{ab}(\omega)\wedge e^c - \frac{\Lambda}{3}e^a\wedge e^b\wedge e^c\right)\epsilon_{abcd} = 0, \tag{3.23}$$

which is recognized as the Einstein field equations in vacuum, that is, equation (2.48). Even when we did not assume any relation between ω and e a priori, the equation of coming from the equation of motion (3.21) establishes the equivalence between tetrad and spin connection, and therefore this establishes the link between tetrad and metric formulation (where torsionless is assumed a priori)[3].

3.2 MacDowell-Mansouri Formulation

The formulation by MacDowell and Mansouri[3] combines the spin connection ω^{ab} and the tetrad e^a as parts of a "Bigger" connection A^{IJ} $(I, J = 0, 1, 2, 3, 4)$ which is a connection of the anti-deSitter group $SO(2,3)$ (AdS). The split goes as follows

$$A^{ab}_\mu = \omega^{ab}_\mu, \quad A^{a4}_\mu = \frac{1}{\ell}e^a_\mu. \tag{3.24}$$

The Lorentz part, corresponds to $a, b = 1, 2, 3$. The constant l is related to the cosmological constant via

$$\frac{\Lambda}{3} = -\frac{1}{\ell^2}. \tag{3.25}$$

We can construct a curvature 2-form associated to the connection A (similar to what we did for the curvature (3.11)) as

$$\mathcal{F}^{IJ}(A) = dA^{IJ} + A^{IK}\wedge A^J_K := \frac{1}{2}\mathcal{F}^{IJ}_{\mu\nu}dx^\mu\wedge dx^\nu, \tag{3.26}$$

where

$$\mathcal{F}^{IJ}_{\mu\nu} = \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + A^I_{\mu K}A^{KJ}_\nu - A^I_{\nu K}A^{KJ}_\mu. \tag{3.27}$$

If we split this curvature into components, it follows that

$$\mathcal{F}^{a4}_{\mu\nu} = \frac{1}{\ell}\left(\partial_\mu e^a_\nu + \omega^a_{\mu b}e^b_\nu - \partial_\nu e^a_\mu - \omega^a_{\nu b}e^b_\mu\right), \tag{3.28}$$

from equation (3.16) we identify this component of the curvature as the torsion $T^a_{\mu\nu}$, therefore

$$\mathcal{F}^{a4}_{\mu\nu} = \frac{1}{\ell}T^a_{\mu\nu}. \tag{3.29}$$

As for the remaining components we have

$$\mathcal{F}^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu + \frac{1}{\ell^2}\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right), \tag{3.30}$$

[3] Alternatively, one can easily check that action (3.19) is equivalent to the usual Einstein-Hilbert action of GR (with cosmological constant)

we identify the first part of the right hand side as the curvature (3.12), therefore

$$\mathcal{F}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \frac{1}{\ell^2}\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right). \tag{3.31}$$

Finally if we define $\Sigma^{ab}_{\mu\nu} := e^a_\mu e^b_\nu - e^a_\nu e^b_\mu$, we can write this as

$$\mathcal{F}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \frac{1}{\ell^2}\Sigma^{ab}_{\mu\nu} := \mathcal{R}^{ab}_{\mu\nu} \tag{3.32}$$

and this is de so called AdS curvature. Therefore, we see that both, torsion and Lorentz curvature are "included" in the "bigger" curvature $\mathcal{F}$. Despite the fact that one cannot use the curvature $\mathcal{F}$ to construct the action for the dynamical theory in four dimensions, it is possible to "break down" the symmetry by projecting the curvature down to Lorentz indices $\mathcal{F} \to \mathcal{R}$. Then, if we propose the action

$$S_{MM}(A) = \kappa \int \mathcal{R}^{ab} \wedge \mathcal{R}^{cd}\epsilon_{abcd} = \kappa \int \left(R^{ab} + \frac{1}{\ell^2}e^a \wedge e^b\right) \wedge \left(R^{cd} + \frac{1}{\ell^2}e^c \wedge e^d\right)\epsilon_{abcd}, \tag{3.33}$$

we see that it splits into the different terms

$$S_{MM} = S_{EC} + S_{CC} + S_E, \tag{3.34}$$

where

$$\begin{aligned} S_{EC} &= \int R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}, \\ S_{CC} &= \int e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd}, \\ S_E &= \int R^{ab} \wedge R^{cd}\epsilon_{abcd}, \end{aligned} \tag{3.35}$$

we immediately recognize the first term as the Einstein-Cartan action, that is, it is the action of GR. The second term is just the cosmological constant term, while, the last term, is the so called Euler invariant, also called the Gauss-Bonnet term. This is a term which does not affect the equations of motion since, when varying the action, one can show, that this term yields $D_\omega R^{ab}$, which vanishes, from the Bianchi identity (3.17). It is customary to write this term in tensor notation, yielding

$$S_{GB} = \int \sqrt{-g}\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right)d^4x. \tag{3.36}$$

The Gauss-Bonnet term, as said before is trivial in 4D, however, it turns to be non-trivial in 4+1D or greater, where its used to construct models known as Gauss-Bonett gravity[30].

Therefore, in this formalism, we can regard GR as a gauge symmetry breaking theory that emerges from the action (3.33). This action is the so called MacDowell-Mansouri action, first proposed by MacDowell and Mansouri[3] as a first attempt to regard gravity as a gauge theory. In fact, more formally, action (3.33) can be written as

$$S_{MM} = \kappa \int \mathrm{Tr}(\hat{F} \wedge \star \hat{F}), \tag{3.37}$$

where $\hat{F}$ is the projection of F into the AdS curvature, and $\star$ denotes the Hodge dual, this is an operator that maps, in n dimensions, $p-$forms into $(n-p)$-forms. In this case, the Hodge dual and the trace are taking into the Lorentz group. Therefore, action (3.37) offers an action from gravity and, at the same time, from a formal point of view, it establishes gravity in a form known from Yang-Mills theories. The reader is encouraged to read references[31, 32] for deeper geometrical and physical aspects on MacDowell-Mansouri theory.

3.2.1 Equations of Motion of the MacDowell-Mansouri Action

In order to simplify calculations, we will like to rewrite the MacDowell-Mansouri action (3.33) in a more slightly convenient form. This First we define the dual of the 2-form $\mathcal{R}$ as

$$\widetilde{\mathcal{R}}^{ab} := \frac{\mathrm{i}}{2} \epsilon^{ab}{}_{cd} \mathcal{R}^{cd}. \tag{3.38}$$

Therefore, the action (3.33) can be written as

$$\begin{aligned} S_{MM} &= \int_{\mathcal{M}} \epsilon_{abcd} \mathcal{R}^{ab}_{\mu\nu} \mathcal{R}^{cd}_{\alpha\beta} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\alpha} \wedge dx^{\beta} \\ &= \int_{\mathcal{M}} \epsilon^{ab}{}_{cd} (2\mathcal{R}^{ab}) \wedge (2\mathcal{R}^{cd}) \\ &= 8\,\mathrm{i} \int_{\mathcal{M}} \mathcal{R}_{ab} \wedge \widetilde{\mathcal{R}}^{ab} \\ &\equiv 8\,\mathrm{i} \int_{\mathcal{M}} \mathcal{R} \wedge \widetilde{\mathcal{R}} \end{aligned} \tag{3.39}$$

Now, we can write this action in terms of the forms R and Σ by replacing $\mathcal{R} = R - \lambda^2 \Sigma$ (and its dual $\widetilde{\mathcal{R}} = \widetilde{R} - \lambda^2 \widetilde{\Sigma}$) in the above action, thus

$$\begin{aligned} S_{MM} &= 8\,\mathrm{i} \int_{\mathcal{M}} (R - \lambda^2 \Sigma) \wedge (\widetilde{R} - \lambda^2 \widetilde{\Sigma}) \\ &= 8\,\mathrm{i} \int_{\mathcal{M}} [R \wedge \widetilde{R} + \lambda^2 (R \wedge \widetilde{\Sigma} - \Sigma \wedge \widetilde{R}) + \lambda^4 \Sigma \wedge \widetilde{\Sigma}]. \end{aligned} \tag{3.40}$$

In order to obtain the equations of motion from the MM action, we consider it in the form (3.39) and take the total variation as follows

$$\begin{aligned} \delta S_{MM} &= 8\,\mathrm{i} \int_{\mathcal{M}} \delta\mathcal{R} \wedge \widetilde{\mathcal{R}} + \mathcal{R} \wedge \delta\widetilde{\mathcal{R}} \\ &= 16\,\mathrm{i} \int_{\mathcal{M}} \delta\mathcal{R} \wedge \widetilde{\mathcal{R}} \end{aligned} \tag{3.41}$$

of course, we are considering the fields e and ω as independent, so we can take the variations with respect to these two fields . Variation with respect to e gives:

$$\begin{aligned}\delta_e S_{MM} &= 16\,\mathrm{i}\int_{\mathcal{M}} \delta_e(R(\omega)-\lambda^2 e\wedge e)\wedge\widetilde{\mathcal{R}} \\ &= -16\,\mathrm{i}\,\lambda^2\int_{\mathcal{M}} \delta_e(e\wedge e)\wedge\widetilde{\mathcal{R}} \\ &= -16\,\mathrm{i}\,\lambda^2\int_{\mathcal{M}} 2\delta_e e\wedge e\wedge\widetilde{\mathcal{R}},\end{aligned} \tag{3.42}$$

since the variation must vanish for arbitrary variation of the tetrad, we must have

$$e\wedge\widetilde{\mathcal{R}} = 0, \tag{3.43}$$

which means that for arbitrary e we must have

$$\widetilde{\mathcal{R}} = 0 \text{ or } \mathcal{R} = 0, \tag{3.44}$$

Therefore, the equation of motion is

$$R = \lambda^2\Sigma. \tag{3.45}$$

In a similar manner we can calculate the variation with respect to ω:

$$\begin{aligned}\delta_\omega S_{MM} &= 16\,\mathrm{i}\int_{\mathcal{M}} \delta_\omega(R(\omega)-\lambda^2 e\wedge e)\wedge\widetilde{\mathcal{R}} \\ &= 16\,\mathrm{i}\int_{\mathcal{M}} \delta_\omega(d\omega+\omega\wedge\omega-\lambda^2 e\wedge e)\wedge\widetilde{\mathcal{R}} \\ &= 16\,\mathrm{i}\int_{\mathcal{M}} (\delta_\omega d\omega+\delta_\omega(\omega\wedge\omega))\wedge\widetilde{\mathcal{R}} \\ &= 16\,\mathrm{i}\int_{\mathcal{M}} d\delta_\omega\omega\wedge\widetilde{\mathcal{R}}+\delta_\omega\omega\wedge\omega\wedge\widetilde{\mathscr{R}}-\delta_\omega\omega\wedge\widetilde{\mathcal{R}}\wedge\omega \\ &= 16\,\mathrm{i}\int_{\mathcal{M}} d\delta_\omega\omega\wedge\widetilde{\mathcal{R}}+\delta_\omega\omega\wedge[\omega,\widetilde{\mathcal{R}}],\end{aligned} \tag{3.46}$$

integrating by parts and using Stokes' theorem we have

$$\int_{\mathcal{M}} d\delta_\omega\omega\wedge\widetilde{\mathcal{R}} = \int_{\partial\mathcal{M}} \delta_\omega\omega\wedge\widetilde{\mathcal{R}} + \int_{\mathcal{M}} \delta_\omega\omega\wedge d\widetilde{\mathcal{R}} = \int_{\mathcal{M}} \delta_\omega\omega\wedge d\widetilde{\mathcal{R}}, \tag{3.47}$$

where we have neglected the surface term. Hence the variation of the action becomes

$$\begin{aligned}\delta_\omega S_{MM} &= 16\,\mathrm{i}\int_{\mathcal{M}} \delta_\omega\omega\wedge d\widetilde{\mathcal{R}}+\delta_\omega\omega\wedge[\omega,\widetilde{\mathcal{R}}] \\ &= 16\,\mathrm{i}\int_{\mathcal{M}} \delta_\omega\omega\wedge(d\widetilde{\mathcal{R}}+[\omega,\widetilde{\mathcal{R}}]) \\ &= 16\,\mathrm{i}\int_{\mathcal{M}} \delta_\omega\omega\wedge D_\omega\widetilde{\mathcal{R}},\end{aligned} \tag{3.48}$$

Therefore, setting the variation equal to zero, the equation of motion we obtain is

$$D_\omega \widetilde{\mathcal{R}} = 0, \tag{3.49}$$

which means

$$D_\omega (\widetilde{\mathcal{R}} - \lambda^2 \widetilde{\Sigma}) = 0. \tag{3.50}$$

On the other hand, we know that the curvature form is given by $R = d\omega + \omega \wedge \omega$, hence

$$\begin{aligned} dR &= d\omega \wedge \omega - \omega \wedge d\omega \\ &= (R - \omega \wedge \omega) \wedge \omega - \omega \wedge (R - \omega \wedge \omega) \\ &= R \wedge \omega - \omega \wedge R \\ &= -[\omega, R], \end{aligned} \tag{3.51}$$

thus, we obtain the Bianchi identity for R:

$$D_\omega R = dR + [\omega, R] = 0, \tag{3.52}$$

similarly $D_\omega \widetilde{R} = 0$ and therefore the equation of motion (3.50) becomes

$$D_\omega \widetilde{\Sigma} = 0. \tag{3.53}$$

3.3 Autodual MacDowell Mansouri Formulation

Let us consider the gravity action proposed in[7]

$$S = \int_{\mathcal{M}} d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} {}^+\mathcal{R}^{ab}_{\mu\nu} {}^+\mathcal{R}^{cd}_{\alpha\beta}. \tag{3.54}$$

In order to define ${}^+\mathcal{R}^{ab}_{\mu\nu}$ it is necessary to define the autodual connection

$${}^+\omega^{ab}_\mu := \frac{1}{2}\left(\omega^{ab}_\mu - \frac{1}{2}\epsilon^{ab}{}_{cd}\omega^{cd}_\mu\right), \tag{3.55}$$

and similarly for ${}^+\Sigma^{ab}_{\mu\nu}$ we have

$${}^+\Sigma^{ab}_{\mu\nu} := \frac{1}{2}\left(\Sigma^{ab}_{\mu\nu} - \frac{1}{2}\epsilon^{ab}{}_{cd}\Sigma^{cd}_{\mu\nu}\right), \tag{3.56}$$

where $\Sigma^{ab}_{\mu\nu} = e^a_\mu e^b_\nu - e^a_\nu e^b_\mu$, as before. With these definitions we are now ready to construct the autodual part of $\mathcal{R}^{ab}_{\mu\nu}$ which reads

$${}^+\mathcal{R}^{ab}_{\mu\nu} = {}^+R^{ab}_{\mu\nu} - \lambda^{2+}\Sigma^{ab}_{\mu\nu}, \tag{3.57}$$

with

$$ {}^{+}R^{ab}_{\mu\nu} = \partial_\mu {}^{+}\omega^{ab}_{\nu} - \partial_\nu {}^{+}\omega^{ab}_{\mu} + {}^{+}\omega^{ja}_{\mu}{}^{+}\omega^{\ \ b}_{\nu j} - {}^{+}\omega^{ja}{}_{\nu}{}^{+}\omega^{\ \ b}_{\mu j}. \tag{3.58} $$

In terms of these new variables, the action (3.54) becomes

$$ S = \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}{}^{+}R^{ab}_{\mu\nu}{}^{+}R^{cd}_{\alpha\beta} - 2\lambda^2 \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}{}^{+}R^{ab}_{\mu\nu}{}^{+}\Sigma^{cd}_{\alpha\beta} + \lambda^4 \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}{}^{+}\Sigma^{ab}_{\mu\nu}{}^{+}\Sigma^{cd}_{\alpha\beta}. \tag{3.59} $$

With some algebra, it can be easily seen that the real part of the last term reduces to

$$ 2\lambda^4 \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\Sigma^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} = 8\lambda^4 \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}e^a_\mu e^b_\nu e^c_\alpha e^d_\beta, \tag{3.60} $$

while the imaginary part yields

$$ 4\lambda^4 \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\eta_{al}\eta_{bm}\Sigma^{al}_{\mu\nu}\Sigma^{bm}_{\alpha\beta}. \tag{3.61} $$

The second term of this action (3.59) is the Ashtekar term while the last one is the cosmological constant term, so this two terms correspond to Ashtekar theory[**?**]. We can see what are the other terms, for instance, the fist term slits in the two terms

$$ \frac{1}{2}\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R^{ab}_{\mu\nu}R^{cd}_{\alpha\beta} + \mathrm{i}\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\eta_{ac}\eta_{bd}R^{ab}_{\mu\nu}R^{cd}_{\alpha\beta}, \tag{3.62} $$

where we have used the identity $\epsilon_{abcd}\epsilon^{ab}{}_{cd} = -2(\eta_{ce}\eta_{df} - \eta_{cf}\eta_{de})$. We identify the fist term as the Euler topological term while the second term is Pontrjagin topological term, hence the fist term is purely topological . Similarly , the second term yields

$$ 2\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} + 2\,\mathrm{i}\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\eta_{ac}\eta_{bd}R^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta}. \tag{3.63} $$

Putting all these results together, the autodual MacDowell-Mansouri action becomes

$$ \begin{aligned} S =& \frac{1}{2}\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R^{ab}_{\mu\nu}R^{cd}_{\alpha\beta} + \mathrm{i}\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\eta_{ac}\eta_{bd}R^{ab}_{\mu\nu}R^{cd}_{\alpha\beta} \\ &- 4\lambda^2\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} - 4\,\mathrm{i}\,\lambda^2\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\eta_{ac}\eta_{bd}R^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} \\ &+ 2\lambda^4\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\Sigma^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} + 4\,\mathrm{i}\,\lambda^4\int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\eta_{al}\eta_{bm}\Sigma^{al}_{\mu\nu}\Sigma^{bm}_{\alpha\beta}. \end{aligned} \tag{3.64} $$

In order to obtain the equations of motion of this action it more convenient to rewrite it in terms of differential forms, and we do this by invoking again the definitions previously established. Hence, making also a redefinition of constants in terms of the constants $\alpha_1, \ldots, \alpha_6$, it is easy to see that the action is written as

$$ \begin{aligned} S =& \,\mathrm{i}\,\alpha_1\int_{\mathcal{M}} R\wedge\widetilde{R} + \mathrm{i}\,\alpha_2\int_{\mathcal{M}} R\wedge R + \mathrm{i}\,\lambda^2\alpha_3\int_{\mathcal{M}}\Sigma\wedge\widetilde{R} + \mathrm{i}\,\lambda^2\alpha_4\int_{\mathcal{M}}\Sigma\wedge R \\ &+ \mathrm{i}\,\lambda^4\alpha_5\int_{\mathcal{M}}\Sigma\wedge\widetilde{\Sigma} + \mathrm{i}\,\lambda^4\alpha_6\int_{\mathcal{M}}\Sigma\wedge\Sigma. \end{aligned} \tag{3.65} $$

Total variation of this action gives

$$\delta S = 4\,\mathrm{i}(\alpha_1+\alpha_2)\int_{\mathcal{M}}\delta R\wedge{}^{+}R + 2\,\mathrm{i}\,\lambda^2(\alpha_3+\alpha_4)\int_{\mathcal{M}}(\delta\Sigma\wedge{}^{+}R+\delta R\wedge{}^{+}\Sigma) + 4\,\mathrm{i}\,\lambda^4(\alpha_5+\alpha_6)\int_{\mathcal{M}}\delta\Sigma\wedge{}^{+}\Sigma, \tag{3.66}$$

where we have used

$$\begin{aligned}
&R\wedge\delta\widetilde{R} = \widetilde{R}\wedge\delta R\\
&\Sigma\wedge\delta\widetilde{\Sigma} = \widetilde{\Sigma}\wedge\delta\Sigma\\
&\Sigma\wedge\delta\widetilde{R} = \widetilde{\Sigma}\wedge\delta R\\
&2^{+}\Sigma = \Sigma+\widetilde{\Sigma}\\
&2^{+}R = R+\widetilde{R}.
\end{aligned}$$

As in the MM case, we can consider the variations with respect to ω and e independently. Variation with respect to ω yields

$$\delta_\omega S = 4\,\mathrm{i}(\alpha_1+\alpha_2)\int_{\mathcal{M}}\delta R\wedge{}^{+}R + 2\,\mathrm{i}\,\lambda^2(\alpha_3+\alpha_4)\int_{\mathcal{M}}\delta R\wedge{}^{+}\Sigma, \tag{3.67}$$

we can set $\alpha_1+\alpha_2 = -(\alpha_3+\alpha_4)/2 \equiv \beta$ without changing the equations of motion, and hence

$$\begin{aligned}
\delta_\omega S &= 4\,\mathrm{i}\,\beta\int_{\mathcal{M}}(\delta R\wedge{}^{+}R - 2\lambda^2\delta R\wedge{}^{+}\Sigma)\\
&= 4\,\mathrm{i}\,\beta\int_{\mathcal{M}}\delta(d\omega+\omega\wedge\omega)\wedge({}^{+}R-2\lambda^{2+}\Sigma)\\
&= 4\,\mathrm{i}\,\beta\int_{\mathcal{M}}d\delta\omega\wedge{}^{+}R+\delta\omega\wedge[\omega,{}^{+}R] - 4\,\mathrm{i}\,\lambda^2\beta\int_{\mathcal{M}}d\delta\omega\wedge{}^{+}\Sigma+\delta\omega\wedge[\omega,{}^{+}\Sigma],
\end{aligned} \tag{3.68}$$

integrating by parts the first terms of each integral we have

$$\begin{aligned}
\int_{\mathcal{M}}d\delta\omega\wedge{}^{+}R &= \int_{\mathcal{M}}\delta\omega\wedge d^{+}R,\\
\int_{\mathcal{M}}d\delta\omega\wedge{}^{+}\Sigma &= \int_{\mathcal{M}}\delta\omega\wedge d^{+}\Sigma,
\end{aligned}$$

which takes the variation $\delta_\omega S$ to

$$\begin{aligned}
\delta_\omega S &= 4\,\mathrm{i}\,\beta\int_{\mathcal{M}}\delta\omega\wedge\left\{d^{+}R+[\omega,{}^{+}R]-\lambda^2(d^{+}\Sigma+[\omega,{}^{+}\Sigma])\right\}\\
&= 4\,\mathrm{i}\,\beta\int_{\mathcal{M}}\delta\omega\wedge(D_\omega{}^{+}R-\lambda^2 D_\omega{}^{+}\Sigma)
\end{aligned} \tag{3.69}$$

hence, $\delta_\omega S = 0$ gives us the equation of motion

$$D_\omega{}^{+}R = \lambda^2 D_\omega{}^{+}\Sigma. \tag{3.70}$$

From Bianchi identities we know that $D_\omega R = D_\omega \widetilde{R} = 0$, and therefore $D_\omega {}^+R = 0$. Therefore the equation of motion reduces to

$$D_\omega {}^+\Sigma = 0. \tag{3.71}$$

Now we take the variation with respect to e

$$\delta_e S = 2\,\mathrm{i}\,\lambda^2(\alpha_3 + \alpha_4) \int_{\mathcal{M}} \delta\Sigma \wedge {}^+R + 4\,\mathrm{i}\,\lambda^4(\alpha_5 + \alpha_6) \int_{\mathcal{M}} \delta\Sigma \wedge {}^+\Sigma, \tag{3.72}$$

setting $\alpha_3 + \alpha_4 = -2(\alpha_5 + \alpha_6) \equiv \gamma$ this becomes

$$\begin{aligned}
\delta_e S &= 4\,\mathrm{i}\,\lambda^2\gamma \int_{\mathcal{M}} \delta\Sigma \wedge (\lambda^{2+}\Sigma - {}^+R) \\
&= 4\,\mathrm{i}\,\lambda^2\gamma \int_{\mathcal{M}} \delta(e \wedge e) \wedge (\lambda^{2+}\Sigma - {}^+R) \\
&= 8\,\mathrm{i}\,\lambda^2\gamma \int_{\mathcal{M}} \delta e \wedge e \wedge (\lambda^{2+}\Sigma - {}^+R),
\end{aligned} \tag{3.73}$$

setting this variation equal to zero yields

$$e \wedge (\lambda^{2+}\Sigma - {}^+R) = 0, \tag{3.74}$$

which means that for arbitrary e we must have

$$\lambda^{2+}\Sigma = {}^+R. \tag{3.75}$$

In summary, the equations of motion of the Autodual MacDowell-Mansouri are

$$\begin{aligned}
D_\omega {}^+\Sigma &= 0 \\
\lambda^{2+}\Sigma &= {}^+R
\end{aligned} \tag{3.76}$$

3.4 Symmetry breaking Proposal on MacDowell-Mansouri Theory

We can propose an action from which one can obtain all the topological terms in 4-dimensions in a natural way, without imposing any additional condition or introducing terms by hand. The action proposed bellow is inspired in the work by Stelle and West [9]. The proposed action is the following

$$S = \int_{\mathcal{M}} d^4x \epsilon^{\mu\nu\alpha\beta} \mathcal{R}^{AB}_{\mu\nu} \mathcal{R}^{CD}_{\alpha\beta} \left(\Phi^E \epsilon_{ABCDE} + \eta_{AB}\eta_{CD} \right), \tag{3.77}$$

where Φ^E is a vector in the internal space, which, in this case we take as $\Phi^E = (0, 0, 0, 0, 1)$. Therefore

$$\mathcal{R}^{AB}_{\mu\nu} \mathcal{R}^{CD}_{\alpha\beta} \Phi^E \epsilon_{ABCDE} = \mathcal{R}^{AB}_{\mu\nu} \mathcal{R}^{CD}_{\alpha\beta} \epsilon_{ABCD4}. \tag{3.78}$$

We define $\epsilon_{ABCD5} \equiv \epsilon_{abcd}$, then

$$\mathcal{R}^{AB}_{\mu\nu}\mathcal{R}^{CD}_{\alpha\beta}\Phi^{E}\epsilon_{ABCDE} = \mathcal{R}^{ab}_{\mu\nu}\mathcal{R}^{cd}_{\alpha\beta}\epsilon_{abcd}. \tag{3.79}$$

The second term in the action can be expanded as

$$\begin{aligned}\mathcal{R}^{AB}_{\mu\nu}\mathcal{R}^{CD}_{\alpha\beta}\eta_{AB}\eta_{CD} &= \mathcal{R}^{AB}_{\mu\nu}\mathcal{R}_{\alpha\beta AB}\\ &= \mathcal{R}^{ab}_{\mu\nu}\mathcal{R}_{\alpha\beta ab} + \mathcal{R}^{a4}_{\mu\nu}\mathcal{R}_{\alpha\beta a4} + \mathcal{R}^{4b}_{\mu\nu}\mathcal{R}_{\alpha\beta 4b} + \mathcal{R}^{44}_{\mu\nu}\mathcal{R}_{\alpha\beta 44}\\ &= \mathcal{R}^{ab}_{\mu\nu}\mathcal{R}_{\alpha\beta ab} + 2\mathcal{R}^{a4}_{\mu\nu}\mathcal{R}_{\alpha\beta a4},\end{aligned} \tag{3.80}$$

moreover, we know that the term $\mathcal{R}^{a4}_{\mu\nu}$ can be identified with the torsion $\mathcal{R}^{a4}_{\mu\nu} \equiv T^{a}_{\mu\nu}$, where

$$T^{a}_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\nu b}e^b_\mu - \omega^a_{\nu b}e^b_\mu, \tag{3.81}$$

therefore

$$\mathcal{R}^{AB}_{\mu\nu}\mathcal{R}^{CD}_{\alpha\beta}\eta_{AB}\eta_{CD} = \mathcal{R}^{ab}_{\mu\nu}\mathcal{R}_{\alpha\beta ab} + 2T^{a}_{\mu\nu}T_{\alpha\beta a}. \tag{3.82}$$

Hence, collecting all the previous terms, the action reads

$$S = \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\left(\mathcal{R}^{ab}_{\mu\nu}\mathcal{R}^{cd}_{\alpha\beta}\epsilon_{abcd} + \mathcal{R}^{ab}_{\mu\nu}\mathcal{R}_{\alpha\beta ab} + 2T^{a}_{\mu\nu}T_{\alpha\beta a}\right). \tag{3.83}$$

we can study the terms in the action separately, for the first term we have (setting $\lambda = 1$)

$$\begin{aligned}S_1 &= \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\mathcal{R}^{ab}_{\mu\nu}\mathcal{R}^{cd}_{\alpha\beta}\epsilon_{abcd}\\ &= \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\left(R^{ab}_{\mu\nu} - \Sigma^{ab}_{\mu\nu}\right)\left(R^{cd}_{\alpha\beta} - \Sigma^{cd}_{\alpha\beta}\right)\\ &= \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\left(R^{ab}_{\mu\nu}R^{cd}_{\alpha\beta} - R^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} - \Sigma^{ab}_{\mu\nu}R^{cd}_{\alpha\beta} + \Sigma^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta}\right),\end{aligned} \tag{3.84}$$

if we wish to rewrite this term in terms of differential forms, following the customary procedure, it is clear that

$$\begin{aligned} d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R^{ab}_{\mu\nu}R^{cd}_{\alpha\beta} &= 4R^{ab}\wedge R^{cd}\epsilon_{abcd},\\ d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} &= 4\theta^a\wedge\theta^b\wedge R^{cd}\epsilon_{abcd},\\ d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\Sigma^{ab}_{\mu\nu}\Sigma^{cd}_{\alpha\beta} &= 4\theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d\epsilon_{abcd}.\end{aligned}$$

Similarly, for the second term in the action we have

$$\begin{aligned}S_2 &= \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\mathcal{R}^{ab}_{\mu\nu}\mathcal{R}_{\alpha\beta ab}\\ &= \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\left(R^{ab}_{\mu\nu} - \Sigma^{ab}_{\mu\nu}\right)\left(R_{\alpha\beta ab} - \Sigma_{\alpha\beta ab}\right)\\ &= \int_{\mathcal{M}} d^4x\epsilon^{\mu\nu\alpha\beta}\left(R^{ab}_{\mu\nu}R_{\alpha\beta ab} - R^{ab}_{\mu\nu}\Sigma_{\alpha\beta ab} - \Sigma^{ab}_{\mu\nu}R_{\alpha\beta ab} + \Sigma^{ab}_{\mu\nu}\Sigma_{\alpha\beta ab}\right),\end{aligned} \tag{3.85}$$

and again, in terms of differential forms, these terms become

$$
\begin{aligned}
d^4x\epsilon^{\mu\nu\alpha\beta}R^{ab}_{\mu\nu}R_{\alpha\beta ab} &= 4R^{ab}\wedge R_{ab},\\
d^4x\epsilon^{\mu\nu\alpha\beta}R^{ab}_{\mu\nu}\Sigma_{\alpha\beta ab} &= 4\theta^a\wedge\theta^b\wedge R_{ab},\\
d^4x\epsilon^{\mu\nu\alpha\beta}\Sigma^{ab}_{\mu\nu}\Sigma_{\alpha\beta ab} &= 4\theta^a\wedge\theta^b\wedge\theta_a\wedge\theta_b,
\end{aligned}
$$

the last term vanishes due to the antisymmetry of the wedge product. Finally, the last term in the action reads in terms of differential forms

$$
d^4x\epsilon^{\mu\nu\alpha\beta}T^{a}_{\mu\nu}T_{\alpha\beta a} = 4T^a\wedge T_a.
$$

Collecting all the terms, the action becomes

$$
\begin{aligned}
S = 4\int_{\mathcal{M}}\epsilon_{abcd}&\left(R^{ab}\wedge R^{cd} - 2\theta^a\wedge\theta^b\wedge R^{cd} + \theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d\right)\\
&+ R^{ab}\wedge R_{ab} - 2\theta^a\wedge\theta^b\wedge R_{ab} + 2T^a\wedge T_a.
\end{aligned}
\tag{3.86}
$$

We see that this action contains the terms: GR, cosmological constant, Euler density, Pontryagin density and the Nieh-Yang density, which arises in a natural way and in its complete form $N_4 = T^a\wedge T_a - \theta^a\wedge\theta^b\wedge R_{ab}$. There are works in which the magnitude of the vector Φ^E is arbitrary, so, a variation with respect to this quantity may be performed and the magnitude fluctuates around a minimum. The importance of this vector lies not only in the fact that it allows to obtain all the topological terms, but it also offers a natural way of breaking the symmetry of the theory. Although this new formulation offers a better way of constructing a full action which, when breaking the symmetry down to Lorentz symmetry, by virtue of inclusion of the vector Φ^E gives GR plus some of the topological invariants in 4D. However, there is another proposal which attempts to construct a full $SO(4,1)-$invariant action with a vector field, which, is not supposed to be constant a priori, but it is an arbitrary vector field, which, once the appropriate conditions are set, the symmetry is broken down so obtaining GR and all the topological invariants in $4D$. These are called the symmetry breaking conditions (SBE). This approach is described in the following chapter.

Chapter 4

Analysis Over the Symmetry Breaking Condition in MacDowell-Mansouri Action

4.1 Considering an Stelle-West type action

MM theory is a gauge theory of gravity with the gauge group $G \supset SO(3,1)$, where G depends on the sign of the cosmological constant Λ. In this work we will consider the case when $\Lambda > 0$, therefore we will use $SO(4,1)$ (the case with $\Lambda < 0$ and $\Lambda = 0$ can be calculated straightforward). Let us take the de Sitter group $SO(4,1)$ as our gauge group, and a 4-dimensional oriented smooth manifold $\mathcal{M}$, and in order to avoid spacetimes with bad causality properties, let us consider $\mathcal{M} = \mathbb{R} \times \Sigma$, Σ is compact and without boundary, and $\mathbb{R}$ represents an evolution parameter. Now choose a principal $SO(4,1)$-bundle P over M. Let t_{AB} be the skew-symmetric generators of the Lie algebra $\mathfrak{so}(4,1)$

$$[t_{AB}, t_{CD}] = f_{ABCD}{}^{EF}\; t_{EF} = \frac{1}{4}\; \eta_{AB,[C}{}^{E}\eta_{D]}{}^{F} t_{EF}, \tag{4.1}$$

where $A, B, \ldots = 0, 1, 2, 3, 5$ and $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$ and $\eta_{AB,CD} = \eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}$. The Cartan-Killing form κ_{ABCD} in the adjoint representation is given by

$$\kappa_{ABCD} = \frac{1}{2}\; \eta_{AB,CD}. \tag{4.2}$$

We define the connection A as a $\mathfrak{so}(4,1)$-valued 1-form on $\mathcal{M}$, $A = A_\mu{}^{AB} t_{AB} dx^\mu$. Then due that $\mathfrak{so}(4,1)$ is a reductive geometry, $\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^{3+1}$, we can make the following identification for the gauge field

$$A_\mu{}^{AB} = \begin{pmatrix} A_\mu{}^{ab} & A_\mu{}^{a5} \\ A_\mu{}^{5b} & 0 \end{pmatrix} = \begin{pmatrix} \omega_\mu{}^{ab} & -\frac{1}{l}\; e_\mu{}^{a} \\ \frac{1}{l}\; e_\mu{}^{b} & 0 \end{pmatrix} \tag{4.3}$$

where ω^{ab} is the well-known spin connection, e^a is the frame field, l a is constant of the dimension of length, introduced for units requirement. The covariant derivative $\boldsymbol{\mathcal{D}}$ of the de Sitter group acts over Lie algebra valued fields $\xi = \xi^{AB} t_{AB}$ as follows

$$\boldsymbol{\mathcal{D}}\xi = \left[D\xi^{ab} + \frac{1}{l} \left(e^a \wedge \xi^{b5} - e^b \wedge \xi^{a5} \right) \right] t_{ab} + 2 \left[D\xi^{a5} - \frac{1}{l}\, e^b \wedge \xi_b{}^a \right] t_{a5}. \tag{4.4}$$

where we have introduced the Lorentz covariant derivative $D\chi^{ab} = d\chi^{ab} + \omega^{ac} \wedge \chi_c{}^b + \omega^{bc} \wedge \chi_c{}^a$.
The field strength, as usual, is defined as $F = dA + \frac{1}{2}\,[A, A]$,

$$F^{AB} = \begin{pmatrix} F^{ab} & F^{a5} \\ F^{5b} & 0 \end{pmatrix} = \begin{pmatrix} R^{ab} - \frac{1}{l^2}\,\Sigma^{ab} & -\frac{1}{l}\,T^a \\ \frac{1}{l}\,T^b & 0 \end{pmatrix} \tag{4.5}$$

where $T^a = De^a = de^a + \omega^a{}_c \wedge e^c$ is the two-form torsion and $R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b$ is the two-form curvature, both are $\mathfrak{so}(3,1)$ valued and, we also have defined, $\Sigma^{ab} = e^a \wedge e^b$.
Stelle and West considered the action

$$I_{MM}\,[A, v] = \int_{\mathcal{M}} \epsilon^{ABCDE} F_{AB} \wedge F_{CD} v_E + \rho(v^E v_E - l^{-2}) \tag{4.6}$$

where v is an arbitrary vector field with dimension of length, ρ is four-form acting as a Lagrange multiplier and l is a real number. We also have defined the totally anti-symmetric tensor field ϵ with $\epsilon^{01235} = +1$ and $\epsilon_{01235} = -1$. The action (4.6) is invariant under $SO(4,1)$, but in order to obtain GR from it, we have to break the $SO(4,1)$ symmetry down to $SO(3,1)$, and this can be made by choosing a preferred direction for the vector field v^E and asking for a subgroup of $SO(4,1)$ which leaves v fixed, we will refer at these two conditions as the symmetry breaking conditions (SBC). For simplicity, let us consider $v^a = 0$ then $v^5 = 1/l$ and the action (4.6) reads

$$I_{MM}\,[A] = \int_{\mathcal{M}} \epsilon^{abcd} F_{ab} \wedge F_{cd}\, \frac{1}{l} = \frac{1}{l} \int_{\mathcal{M}} \epsilon^{abcd} \left(R_{ab} \wedge R_{cd} - \frac{1}{l^2}\, R_{ab} \wedge \Sigma_{cd} + \frac{1}{l^4}\, \Sigma_{ab} \wedge \Sigma_{cd} \right) \tag{4.7}$$

where ϵ^{abcd} is the Levi-Civita tensor in $\mathfrak{so}(3,1)$, i.e., $\epsilon^{0123} = +1$. From the last action, we can recognize the Euler class plus the Palatini action of GR with non-vanishing cosmological constant term ($\frac{1}{l^2} = \frac{\Lambda}{3}$). As we can observe, (4.6) is essentially GR since the variation of the topological term vanishes due to the Bianchi identity. let us introduce a more general action that involves more topological terms as the Nieh-Yan term and the second Chern class in $SO(4,1)$. To do so, let us define a pseudo-projector for $\mathfrak{so}(4,1)$ as follows

$$\Pi^{ABCD} \xi_{CD} = \frac{1}{2} \left(\alpha \eta^{AB,CD} + \epsilon^{ABCDE} v_E \right) \xi_{CD} = \frac{1}{2} \left(\alpha \eta^{AB,CD} + \widetilde{\epsilon}^{ABCD} \right) \xi_{CD} = \widetilde{\xi}^{AB} \tag{4.8}$$

where α is a constant with units of $length^{-1}$. The main idea in defining (4.8) is to construct the action as some kind of pure connection (anti) self-dual formulation for gravity, inspired by the formulation of Plebański and the Ashtekar's canonical formulation of gravity.
We consider the action

$$I_{GMM}\,[A, v] = \int_{\mathcal{M}} \prec \widetilde{F} \wedge \widetilde{F} \succ + \rho(v^E v_E - \ell^{-2}) \tag{4.9}$$

where $\prec \quad , \quad \succ$ is the trace over the Lie algebra generators. Written in this form and since the metric structure is not present explicitly through the frame field, this approach is closer to the (Anti) self-dual pure connection formulation of gravity given by Capovilla, Dell and Jacobson. That formulation we have a simple solution for the equations of motion, and this solution, known as the instanton solution, is given by the (anti) de Sitter or even flat space-time ($F = 0$).Then by considering the definition of the pseud-projector, the action (4.9) can be rewritten as

$$I_{GMM} = \int_{\mathcal{M}} (\alpha^2 - v^E v_E) F^{AB} \wedge F_{AB} + \alpha \widetilde{\epsilon}^{ABCD} F_{AB} \wedge F_{CD} - 2F^{AC} \wedge F_C{}^B v_A v_B + \rho (v^E v_E - l^{-2}). \quad (4.10)$$

Now, let us consider the equation of motion associated with ρ together with the SBC, then the action reads

$$\begin{aligned} I_{GMM}[e,\omega] = \frac{2\Lambda\gamma}{3} \int_{\mathcal{M}} & \frac{\gamma^2 - 1}{2\gamma} F^{AB} \wedge F_{AB} + 2\epsilon^{abcd} R_{ab} \wedge R_{cd} + d\left(\frac{1}{\gamma}\, e^a \wedge T_a\right) \\ & - \left(\epsilon^{abcd} R_{ab} \wedge \Sigma_{cd} - \frac{1}{\gamma} R^{ab} \wedge \Sigma_{ab} - \frac{\Lambda}{6}\, \epsilon^{abcd} \Sigma_{ab} \wedge \Sigma_{cd}\right). \end{aligned} \quad (4.11)$$

where we have defined $\alpha = 1/(l\gamma)$ and γ is the Barbero-Immirzi (BI) parameter, which is a nonzero dimensionless parameter. We have separated the action into the topological sector, which correspond to the firs three terms in the action: the second Chern class, the Euler class and the Nieh-Yan Class, and the dynamical sector, which corresponds to the last three terms: the Palatini action, the Holst modification and the cosmological constant sector. It is important to mention that the Holst modification is not topological neither dynamical, since its its variation is non-zero, and it cannot be rewritten as total divergence, and even more, at the classical level doesn't affect the dynamical behavior of the theory, but we associate it to the dynamical sector due to its apparent effects at the quantum level since it appears at the prediction of the spectra of geometrical operators, area and volume, and consequently appearing in the black hole entropy calculation, even more it labels inequivalent quantizations in the framework of kinematical loop quantum gravity and modifies the symplectic structure which can not be unitarily implementable at quantum level.
On the other hand, the metric tensor is constructed by using the covariant derivative on v^A

$$g_{\mu\nu} = e_\mu{}^a e_{\nu a} = \boldsymbol{\mathcal{D}}_\mu v^A \boldsymbol{\mathcal{D}}_\nu v_A \quad (4.12)$$

where we have used the SBC and the fact that the metric tensor is defined upon a constant factor. Then as we can observe, the tetrad field is given essentially in function of the vector field v, but even more, from the action I_{GMM} is possible to obtain as equation of motion, a well-known equation, the zero torsion condition that let us write the spin-connection as a function of the tetrad field and so, the spin connection as a function of the vector field v. Finally, the fundamental fields in this framework, the tetrad field and the spin connection, i.e., the components of the $\mathfrak{so}(4,1)$-valued connection A are given as functions of v. Being the vector field v a main ingredient that let us view gravity as a gauge theory of $SO(4,1)$, then it is important to find why it arises is fundamental.

4.1.1 Equations of motion without SBC

Let us start by considering the action in eq.(4.9)

$$I_{GMM}[A,v] = \int_{\mathcal{M}} \prec \widetilde{F} \wedge \widetilde{F} \succ, \tag{4.13}$$

where we have taken into account the equation of motion from ρ, i.e., $v^2 = l^{-2}$. This action can not be written as a total derivative therefore this is not a topological action. Since v is not necessarily a constant vector, it implies, by taking its covariant derivative that

$$\boldsymbol{\mathcal{D}}(v^A v_A) = \boldsymbol{\mathcal{D}}(l^{-2}) \quad \Rightarrow \quad v^A \boldsymbol{\mathcal{D}} v_A = 0 \quad \Rightarrow \quad dv^4 = -\frac{1}{v^4} v^a dv_a, \tag{4.14}$$

where we have written $v^A = (v^a, v^4)$ and $v^4 \neq 0$. The equations of motion from I_{GMM}, are given by

$$\begin{aligned} \epsilon^{ABCDE} F_{AB} \wedge F_{CD} &= \frac{4}{\alpha}\, F^{EA} \wedge F_A{}^B v_B, & (4.15)\\ \epsilon^{ABCDE} F_{CD} \wedge \boldsymbol{\mathcal{D}} v_E &= \frac{2}{\alpha}\, F^{C[A} \wedge \boldsymbol{\mathcal{D}}(v^{B]} v_C), & (4.16) \end{aligned}$$

where the former equation is related to the variation with respect to v, and the second with respect to A. The equation of motion coming from A implies a dynamical equation for v, then our task is to find some solutions for this dynamical behavior which led us to understand the role played by v in a more fundamental level. In the next subsections we will present two different dynamical solution for v.

4.1.2 Case $\boldsymbol{\mathcal{D}} v^E = A^{E4} v_4$

Let us consider the case $\boldsymbol{\mathcal{D}} v^E = A^{E4} v_4$, with v^4, non-vanishing at least for one component of v. This gauge implies the following equations

$$\begin{aligned} v^A \boldsymbol{\mathcal{D}} v_A &= \frac{1}{l}\, v^a e_a v^4 = 0, & (4.17)\\ Dv^a &= 0, & (4.18)\\ dv^4 &= -\frac{1}{l}\, v^a e_a. & (4.19) \end{aligned}$$

From eq.(4.17) we observe two different cases, one of them is that v^4 vanishes, but from eq.(4.19) we see that $v^a e_a$ vanishes as well, however since we are considering the case with non-degenerate tetrad field then v^a also vanishes, then $v^E = 0$ which it's the trivial case that we are not considering. In view of this, let us consider the second case, let v^4 be non-vanishing (at least some finite points at the manifold). Nonetheless, as $v^4 \neq 0$, then we have again the case $v^a e_a = 0$, implying that $v^a = 0$ since we are considering non-degenerate tetrad fields and from eq.(4.19), we observe that v^4 is constant, and equal to l^{-1} due to the algebraic constraint imposed over $v \cdot v$, and as we remember, this is just the case we considered when the SBC was taken into account.
Our new step is to consider equations eq.(4.15) and eq.(4.16), and substitute back $v^E = (0,0,0,0,l^{-1})$

together with our gauge condition, yielding the following equations

$$(\gamma\eta^{ab,cd} + \epsilon^{abcd})T_c \wedge e_d = 0, \tag{4.20}$$

$$\epsilon^{abcd}R_{bc} \wedge e_d - \frac{\Lambda}{3}\epsilon^{abcd}e_b \wedge e_c \wedge e_d = \frac{1}{\gamma}DT^a, \tag{4.21}$$

the first equation is the well-known zero torsion condition which let us write the spin connection as a function of the tetrad field, the second one is the Einstein equations in vacuum. This pair of equations come directly from the condition in eq.(4.16).
From eq.(4.15) and as well as eq.(4.16), we obtain

$$\frac{\Lambda^2}{9} = \frac{\epsilon_{abcd}R^{ab} \wedge R^{cd}}{\epsilon_{abcd}\ e^a \wedge e^b \wedge e^c \wedge e^d}, \tag{4.22}$$

which provides an algebraic equation for the cosmological constant that can allow us to fix its value [33, 34]. So the gauge given on shell in this subsection, over the covariant derivative of v, not only gives us GR with cosmological constant, but even more, it fixes the value of the cosmological constant, too. In some sense, this gauge is almost the same as breaking the symmetry group from the beginning, but it gives us more information by consistently deliver us the value of the cosmological constant, so Λ is calculated within the theory, rather than being a free parameter that has to be measured.

4.1.3 Case $\boldsymbol{\mathcal{D}}v^E = 0$

This case is particularly interesting since we cannot construct the metric by considering equation (4.12), as in the previous cases. Im this case we obtain the equations

$$Dv^a = \frac{1}{l}\ e^a v^4, \tag{4.23}$$

$$dv^4 = -\frac{1}{l}\ v^a e_a, \tag{4.24}$$

from eq.(4.14) and eq.(4.24) we obtain that the one-form tetrad field is completely defined by the components of the vector field v and its derivatives, as follows

$$e^a = \frac{l}{v^4}\ dv^a. \tag{4.25}$$

From the last equation and eq.(4.23) we obtain

$$\omega^a{}_b v^b = 0, \tag{4.26}$$

which tell us that the spin connection must be orthogonal to the Minkowski components of v. On the other hand, by applying the $\mathfrak{so}(3,1)$ covariant derivative to equation eq.(4.23), we obtain

$$v^4 De^a = 0 \quad \Rightarrow \quad De^a = 0, \tag{4.27}$$

where we have considered $F^{AB}v_B = 0$ (this is so by taking the $\mathfrak{so}(4,1)$ bundle covariant derivative to the gauge), and also, in general $v^4 \neq 0$. Then, as well as in previous cases, we find the torsion condition, this equation let us write the spin connection as function of the tetrad field, but since the tetrad field depends on v, so ω. Finally we obtain

$$\omega = \omega(e) = \omega(v^a, v^5) \qquad\qquad e = e(v^a, v^5). \tag{4.28}$$

Then, as we can observe, all the fundamental objects are given by the vector field v^A, so this vector let us construct the frame bundle and (locally) the $SO(3,1)$-bundle, where the spin connection must be orthogonal to the vector v^a.
Now, from $F^{AB}v_B = 0$ and eq.(4.26), we obtain

$$\chi^{ab}v_b = 0 \qquad \text{where} \qquad \chi^{ab} = d\omega^{ab} - \frac{1}{l^2}\,\Sigma^{ab}. \tag{4.29}$$

Let us note that if $det|\chi| \neq 0$ then $v^a = 0$ so the component v^4 is constant and the tetrad field and spin connection vanishes identically, so the theory gives no useful information. So we have two different cases, one of them is by considering the determinant $det|\chi| = 0$ and calculating the null vectors v^a (this case goes beyond the scope of this work), and the second case is by considering $\chi^{ab} = 0$, this case implies $l^2 d\omega^{ab} = \Sigma^{ab}$. Thus, by taking into account this last case and the zero torsion condition, we observe that the dynamical equation coming from the equations of motion for v is

$$\epsilon^{abcd}F_{ab}\wedge F_{cd} = 0 \qquad \Rightarrow \qquad \epsilon^{abcd}\omega_a{}^f \wedge \omega_{fb}\wedge \omega_c{}^g \wedge \omega_{gd} = 0, \tag{4.30}$$

this equation vanishes identically, so we do not have any dynamical equation of motion hence this gauge together with the condition $\chi = 0$ are requirements related to topological field theories. Then as we can observe, v parametrizes a family of topological torsionless field theories. For an e-print of the work described in this section see [10]. On the other hand, recently attention has been paid to the fact that some results indicate that some models of gravity in three and four dimensions seem to emerge from certain theories of higher dimensionality. In particular, under the scheme of the so-called Topological M-Theory, it has been possible to obtain gravity actions in three and four dimensions[35, 36]. In this approach, an action constructed as a volume functional of a 7-dimensional manifold is considered. This functional is known as the Hitchin Functional and is constructed from "stable" 3-forms. from this point of view, all relevant information is contained in the 3-forms. When considering an appropriate ansatz for the 3-form and performing a dimensional reduction of the theory to 4 or 3 dimensions, the resulting equations of motion coming from the equations of motion of Toplogical M-Theory are those of $3D$ gravity, or those of 4D self dual gravity. In particular, it was shown that the equations of motion of Topological M-Theory reduce to the equations of motion (3.76). Therefore it is interesting to study whether it is possible to obtain more general gravity actions such as those described in this chapter from theories of higher dimensionality such as Topolgical M-Theory. With this we could see, for example, if the SBC described in section in the action 4.9 arise as a consequence of purely geometric property of the $7D$ manifold, among other things. This is being worked on and results could be reported in the future.

Chapter 5

Entropic Approach to Gravity

5.1 Thermodynamics of Space Time

In his celebrated work[11], Jacobson first established the relation between thermodynamics and General Relativity by demanding that the fundamental thermodynamical relation $\delta Q = TdS$ holds through each spacetime point, interpreting δQ and T as the energy flux and Unruh temperature seen by an accelerated observer just inside the so called Rindler horizon. Viewed in this way, the Einstein field equations emerge as an equation of state of spacetime. In order to understand the main ideas of Jacobson's approach let us briefly describe his procedure in this section. For more details about the derivation, the reader is encouraged to read the main reference.

Jacobson's ideas can be illustrated by the thermodynamics of a simple homogeneous system. By knowing the entropy function $S(E, V)$ one can obtain the equation of state from the relation $\delta Q = TdS$. For instance, in the case of weakly interacting molecules, one has that the entropy $S(E, V) = \ln \Omega$, where Ω is the number of accessible states. For an ideal gas this entropy can be written as $S \sim \ln V + f(E)$ for some function $f(E)$. This gives $\frac{\partial S}{\partial V} \sim V^{-1}$, and we know that $\frac{\partial S}{\partial V} = \frac{P}{T}$, therefore $PV \sim T$, which is the equation of state for an ideal gas.

In spacetime dynamics, one defines heat as the energy that flows across a causal horizon and it can be detected via the gravitational field it generates, but one cannot know its particular nature from outside the horizon. This horizon is not necessary a black hole event horizon, It can be simply the boundary of the past of any observer $\mathcal{O}$. In addition, as known from previous results[37], entropy is assumed to be proportional to the area of the horizon. The temperature of the system is related to the Unruh temperature $T = \hbar\kappa/2\pi$ as observed from the perspective of a uniformly accelerated observer inside the horizon. In this expression the temperature, κ is the so called surface gravity. The same observer should measure the energy flux defining the heat flow. One important fact of the procedure is the following: Note that, until now, the system has been considered as any causal horizon. However, such system, in general, is not in equilibrium, since the horizon is not static; it is expanding, contracting, or shearing, therefore, equilibrium thermodynamics cannot be applied is this way, hence, it is necessary to impose the equivalence principle, that is, a small neighborhood of each spacetime point p is seen as a piece of flat spacetime. Through p a small 2-surface element $\mathcal{P}$ is considered such that it has vanishing expansion and shear at p. It is always possible to choose $\mathcal{P}$

through p so that the expansion and shear vanish in a first order neighborhood of p. the past horizon of such a region $\mathcal{P}$ is called the "local Rindler horizon of $\mathcal{P}$", in other words, this local Rindler horizon is instantaneously stationary, or, in "local equilibrium" at the point p.

We know consider any local Rindler horizon through a spacetime point p. Let χ^a be any be an approximate local boost Killing field generating this horizon, with the direction of χ^a chosen to be future pointing to the "inside" past of the horizon. The heat flux is given by

$$\delta Q = \int_{\mathcal{H}} T_{ab}\chi^a d\Sigma^b, \tag{5.1}$$

If k^a is the tangent vector to the horizon generators for an affine parameter λ that vanishes at $\mathcal{P}$ and is negative to the past of $\mathcal{P}$, then $\chi^a = -\kappa\lambda k^a$, and $d\Sigma^a = k^a d\lambda d\mathcal{A}$, where $d\mathcal{A}$ is the area element on a cross section of the horizon. Thus we can rewrite the heat flux as

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{ab} k^a k^b d\lambda d\mathcal{A}. \tag{5.2}$$

If now we take into consideration that the entropy is proportional to the horizon area, $dS = \eta\delta\mathcal{A}$, where η is an undetermined dimensional constant. The area variation is given by

$$\delta\mathcal{A} = \int_{\mathcal{H}} \theta d\lambda d\mathcal{A}, \tag{5.3}$$

where θ is the expansion of the horizon generators.

Recalling that the local Rindler horizon has vanishing expansion, this vanishing expansion expansion must appear just the right rate so that the area increase of a portion of the horizon will be proportional to the energy flux across it. This requirement imposes the following condition on the curvature

$$\theta = -\lambda R_{ab} k^a k^b, \tag{5.4}$$

where R_{ab} is the Ricci tensor. Therefore, the fundamental relation $\delta Q = TdS = (\hbar\kappa/2\pi)\eta\delta\mathcal{A}$ establishes

$$\kappa \int_{\mathcal{H}} \lambda T_{ab} k^a k^b d\lambda d\mathcal{A} = \frac{\hbar\kappa}{2\pi}\eta \int_{\mathcal{H}} \lambda R_{ab} k^a k^b d\lambda d\mathcal{A}, \tag{5.5}$$

which is valid only if

$$T_{ab} k^a k^b = \frac{\hbar\eta}{2\pi} R_{ab} k^a k^b. \tag{5.6}$$

The latest equation implies that

$$\frac{2\pi}{\hbar\eta} T_{ab} = R_{ab} + f g_{ab} \tag{5.7}$$

for some function f. Moreover, local conservation of energy $\nabla T = 0$ implies that

$$f = -\frac{R}{2} + \Lambda, \tag{5.8}$$

for some constant Λ. Therefore, finally we obtain

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta}T_{ab}, \tag{5.9}$$

if we define the Newton's graviational constant as $G = (4\hbar\eta)^{-1}$ we finally obtain

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}, \tag{5.10}$$

which we immediately recognize as the Einstein field equations (Non vacuum version of (2.52) with cosmological constant. It is important to recall that this thermodynamic derivation of the Einstein equation of state Jacobson employed the existence of local equilibrium conditions. The relation $\delta Q = TdS$ only applies to variations between nearby states of local thermodynamic equilibrium. For instance, in free expansion of a gas, entropy increase is not associated with any heat flow, and this relation is not valid. Moreover, local temperature and entropy are not even well defined away from equilibrium. In the case of gravity, the systems were defined by local Rindler horizons, which are instantaneously stationary, in order to have systems in local equilibrium. In this sense, as pointed out by Jacobson: given local equilibrium conditions, the Einstein equation gives a system of local partial differential equations whose solutions include propagating waves. This can be seen as something analogous to sound in a gas propagating as an adiabatic compression wave which is a traveling disturbance of local density, and it should not be canonically quantized as if it were a fundamental field, even though the individual molecules are quantum mechanical. By analogy, this suggests that it may not be correct to canonically quantize the Einstein equations, even if they describe a phenomenon that is ultimately quantum mechanical.

5.2 Newtonian Gravity as an Entropic Force

In this section we will review the work by Verlinde[12] and describe the main ideas on the entropic origin of gravity. An entropic force is an effective macroscopic force that has its origins in a system due to the tendency of entropy to increase, and no fundamental fields are associated with this force, which is expressed in terms of entropy gradients and is totally independent of the microscopic dynamics. Entropic forces are found in systems such as colloids or biophysical systems, in particular, osmosis is a phenomenon caused by entropic forces. The statistical tendency to return to a configuration of maximum entropy is translated into a macroscopic force, points in the direction of increasing entropy. This force is known as entropic force. Entropic forces turn to be proportional to the temperature, and, at the macroscopic level an entropic force can be conservative as long as the temperature is kept constant. In the case of osmosis across a semi-permeable membrane which carries a temperature T, a particle 'entropically' compelled on one side of the membrane experiences and effective force equal to

$$F\Delta x = T\Delta S, \tag{5.11}$$

and this is the entropic force, and we see that, in order to have a non zero force, we need a non zero temperature. This expression for the entropic force is fundamental in order to derive the Newtonian force as an entropic force. Verlinde's idealization of gravity features a conception of space as a "device" that stores information. That is, for example, the information on the movement of a particle (position, momentum, etc.) is stored in space, and of course, the ability to store information in a certain region of space is finite. In this sense, it is assumed that the information is stored on surfaces called "screens" and this information is stored in discrete bits. Moreover, the dynamics of the screens are determined by some rules related to the way the information is being processed on the screens, With these ideas, and motivated by Bekenstein's work[38], one postulates that the change of entropy associated with the information on the screens is given by

$$\Delta S = 2\pi k_B \quad \text{when} \quad \Delta x = \frac{\hbar}{mc}, \tag{5.12}$$

where Δx is the separation between the particle and the screen. By assuming that the change of entropy is linear in the displacement Δx this can be rewritten as

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x. \tag{5.13}$$

The change of entropy is also linear in the mass m since, if the consider a splitting of the particle into two or more lighter sub-particles, each of them will carrying its own associated change of entropy after a shift Δx, and we know that mass and entropy are additive, therefore it is natural that the change of entropy is proportional to the mass as well. Now main the question is still unanswered: How does force emerge? the key idea is to assume that the force is given by the entropic force (5.11). Now, what is the temperature T? From Newtonian mechanics, we know that a force leads to an acceleration, and, we know from Unruh's works[39] that acceleration and temperature are intimately related via the Unruh effect: an observer in an accelerated frame experiences a temperature

$$k_B T = \frac{1}{2\pi} \frac{\hbar a}{c}, \tag{5.14}$$

where a is the acceleration. Now we consider that this is the temperature associated with the bits on the screen. Note that doing so, from equations (5.13) and (5.11) we obtain Newton's law

$$F = ma. \tag{5.15}$$

It is important to mention that, in this case, equation (5.14) must be interpreted as the temperature required to cause an acceleration a, and not as the temperature caused by an acceleration. If now we consider that the surface is a closed surface, concretely a sphere. The maximal storage space will be proportional to the area A. That is, If we denote the number of bits on the surface by N, we write

$$N = \frac{Ac^3}{G\hbar}. \tag{5.16}$$

The constant G is not a priori assumed to be Newton's gravitation constant. Now suppose that the energy present in the system is E, then, by the equipartition rule we have

$$E = \frac{1}{2} N k_B T, \tag{5.17}$$

from which we can determine the temperature. In addition, we can relate this energy with the mass that would emerge in the part of space enclosed by the surface by

$$E = Mc^2, \tag{5.18}$$

this mass may not be visible in the emerged space, but we would note its presence through its energy. Then, from the equations above, we obtain

$$Mc^2 = \frac{1}{2} N k_B T, \tag{5.19}$$

using the equations for the number of bits and the temperature (5.14) and (5.16) we get

$$M = \frac{aA}{4\pi G}, \tag{5.20}$$

finally, inserting $A = 4\pi R^2$ and $a = F/m$, and solving for F, we obtain the familiar law

$$F = G\frac{Mm}{R^2}, \tag{5.21}$$

and we have recovered Newton's law of gravitation, practically from first principles. In this sense, gravitational force is seen as an entropic force. For deep comments on the naturalness and robustness of this derivation, and more details see[12].

According to a work by Andrew Randono[13], by considering that the entropy related to the screens is not the typical entropy-area relationship found by Bekenstein, but a generic modification to this law given by

$$S[A] = \frac{Ac^3 k_B}{4\hbar G} + k_B \mathfrak{s}(A). \tag{5.22}$$

one can follow Verlinde's approach (apart from some details in the derivation) and obtain deviations of the Newtonian gravitational force (5.21) which arise from the deviations of the entropy from the area law. The considerations needed to derive this modified force are very similar to those in Velrinde's derivation. In consonance with this work, the corresponding force is given by

$$F = -\frac{GMm}{R^2}\left[1 + 4\ell_P^2 \frac{\partial s}{\partial A}\right]_{A=4\pi R^2}, \tag{5.23}$$

where the first term corresponds to the usual area law and the second term, $\mathfrak{s}(A)$, includes the other contributions to the entropy. This formula will be of great importance for us in the following section, since it will used to derive a modified Newtonian force, by using a specific form for the corrected entropy-area relationship. This derivation will be described int he following chapter.

Chapter 6

Some New Results in Entropic Gravity

6.1 Obtaining an Emergent Modified Newtonian Gravity

Verlinde's ideas have inspired several models that attempt to modify Newtonian gravity. This has been achieved by proposing quantum modification to the entropy or by using new definitions of entropy [40] to find modifications to Newton's gravitational force and in some cases modifications to gravity in the cosmological scenario [27]. As mentioned before, We can write a generic modification to the entropy-area relationship according to (5.22) with te resulting modified force given by (5.23). With this in mind, let us find the emergent modified Newtonian force related with the area-entropy derived in Appendix A. This entropy-area relationship reads

$$\frac{S[A]}{k_B} = (1+\Delta(\epsilon))\frac{A}{4l_P^2} - \frac{1}{2}\ln\frac{A}{4l_P^2} + \Gamma(\epsilon)\left(\frac{A}{4l_P^2}\right)^{1/2} + \epsilon\left(\frac{A}{4l_P^2}\right)^{3/2}. \tag{6.1}$$

The modifications to the entropy of the proposed supersymmetric Schwarzschild model can be understood as follows: The first term is the usual Bekenstein-Hawking entropy, the logarithmic correction seems to be a universal correction and has been derived in different approaches in the study of black holes [41–44]. This volumetric correction is physically natural since one expects that the ordinary degrees of freedom of quantum fields scale as the volume instead of the area. Therefore, this correction can be interpreted as a transition between the holographic phase of gravity and the field theoretical phase. Different LQG-inspired models of black hole entropy derive precisely a correction of this type, for example see [45], which result suggests to study the entropy correction of the form $s(A) \sim k_B \left(Ac^3/G\hbar\right)^{3/2}$. Moreover, a correction of this type has been also considered in [13]. In our work this term will play a fundamental role, and an important feature of our work is that we do not just propose this correction to the entropy-area law, but we obtain it from a supersymmetric minisuperspace model, which we widely discuss in appendix A. The third term is proportional to a linear length, this term can be interpreted as an effective contribution due to a self-gravitating gas, this behavior in the scaling is consistent with the entropy calculated for a self-gravitating gas where $S \sim V^{1/3}$ [46].

In order to proceed, let us split the entropy (6.1) in accordance to Eq.(5.22), and the calculation for the modified Newtonian force follows straightforwardly from (5.23). For our entropy, the resulting modified force is

$$\mathbf{F}_M = -\frac{G_{eff}Mm}{R^2}\left[1+\frac{3\sqrt{2\pi}}{l_P(1+\Delta(\epsilon))}\epsilon R+\frac{l_P\Gamma(\epsilon)}{2\sqrt{\pi}(1+\Delta(\epsilon))}\frac{1}{R}-\frac{l_P^2}{2\pi(1+\Delta(\epsilon))}\frac{1}{R^2}\right]\hat{\mathbf{R}}, \quad (6.2)$$

where $G_{eff} = (1+\Delta(\epsilon))G$ is an effective gravitational constant. In this force we identify We see that we recover the usual Newtonian force in the limit $\epsilon \to 0$ modulo a term $\mathcal{O}(R^{-4})$ which can be neglected since it is suppressed by l_p^2. In this force we identify the following corrections: the term proportional to R^{-1} corresponding to the volumetric correction of the entropy, the term proportional to R^{-3} corresponding to the linear correction to the entropy and finally the term proportional to R^{-4} corresponding to the logarithmic correction. The implications of these correction terms, in particular, the term arising from the volumetric correction in the entropy, will become clearer bellow.

We can obtain the modified gravitational potential by integrating the force (6.2). Up to an arbitrary integration constant σ the potential yields

$$\Phi_M = -\frac{G_{eff}M}{R}\left[1-\frac{3\sqrt{2\pi}}{l_P(1+\Delta(\epsilon))}\epsilon R\ln\frac{R}{\sigma}+\frac{l_P\Gamma(\epsilon)}{4\sqrt{\pi}(1+\Delta(\epsilon))}\frac{1}{R}-\frac{l_P^2}{6\pi(1+\Delta(\epsilon))}\frac{1}{R^2}\right]. \quad (6.3)$$

The effective matter density such that the Poisson equation $\nabla^2\Phi_M = 4\pi\rho_{eff}$ is satisfied is straightforwardly calculated, resulting in

$$\rho_{eff} = \frac{GM}{2\pi R}\frac{l_P^2}{2\pi R^4}\left[1-\frac{\sqrt{\pi}}{2l_P}\Gamma(\epsilon)R+\frac{3}{2}\left(\frac{\sqrt{2\pi}}{l_P}\right)^3\epsilon R^3\right]. \quad (6.4)$$

We can have different interpretations to this modified matter density, one can consider that the origin of ρ_{eff} is a consequence to the presence of a point mass M and through some unknown process gives ρ_{eff}. The other possibility is to consider that we have a point particle of mass M that generates the potential Φ_M by using the appropriate limit of some unknown theory of gravity.

In order to analyze the consequences of the modifications in the force (6.2) let us consider a particle of mass m in circular motion with the radius of the orbit being R. According to Eq.(6.2), the velocity of the particle is given by

$$\begin{aligned}\frac{mv^2}{R} &= \frac{G_{eff}Mm}{R^2}\left[1+\frac{3\sqrt{2\pi}}{l_P(1+\Delta(\epsilon))}\epsilon R\right. \\ &+ \left.\frac{l_P\Gamma(\epsilon)}{2\sqrt{\pi}(1+\Delta(\epsilon))}\frac{1}{R}-\frac{l_P^2}{2\pi(1+\Delta(\epsilon))}\frac{1}{R^2}\right].\end{aligned} \quad (6.5)$$

We can see that for large R, the second term in (6.5), corresponding to the entropic volumetric correction is the leading term. Concretely, in the limit of large R the velocity results in

$$v^2 \approx \frac{3\sqrt{2\pi}GM}{l_P}\epsilon, \quad (6.6)$$

We can relate this example with the case of some star in circular motion around the center of a galaxy, and we can investigate the consequences of the modification to the Newtonian force on its rotation velocity. For this purpose, let us make use of one of the main ideas of the Modified Newtonian dynamics[15]. Roughly speaking, MOND is a one parameter phenomenological theory that reproduces the observed galactic rotation curve. It is of particular interest the behavior in the limit of large R, since in this limit the obtained rotation velocity is a constant, which is in accordance with the observations, and this behavior is not predictable by ordinary Newtonian Dynamics. In MOND, the rotation velocity for large R is given by

$$v^2 = \sqrt{GMa_0}, \tag{6.7}$$

where a_0 is the characteristic MOND acceleration $a_0 \approx 10^{-10} m/s^2$. At this point, we observe that our model bears one free parameter ϵ, and, the idea is that, since for large R our model yields a constant velocity, such as in MOND, then we can exploit this fact in order to relate our parameter ϵ with the characteristic quantities of MOND, so that we can reproduce the observed galactic rotations curves. Thus, comparing (6.6) with (6.7) we can see that, in order to both results to agree we must have

$$\epsilon = \frac{1}{6\pi}\sqrt{\frac{a_0 h}{Mc^3}}. \tag{6.8}$$

With this, we can obtain an approximate value for ϵ by using the characteristic MOND mass $M \approx 10^{58} kg$. This gives

$$\epsilon \approx 10^{-58}. \tag{6.9}$$

Of course, in order to obtain a more accurate value for our parameter, it is necessary a deeper analysis by comparing our model with actual observational data on galactic rotation curves. Nonetheless, it is worth mentioning that in a recent work this model has been used to study the anomalous precession of the perihelion of Mercury[47]. In this work, in order to have the right precession for the orbit of Mercury the authors find the bound

$$\epsilon \leq 7,755 \times 10^{-58}. \tag{6.10}$$

This result strengths the former prediction for the value of ϵ in a more accurate way, however, as mentioned before, further analysis is needed.

6.2 Cosmological Scenario: A Plausible Entropic Origin of the Cosmological Constant Λ

As it is well known, the Friedmann equations are a set of equations in cosmology that govern the dynamics of the universe assuming homogeneity and isotropy at large scales. These equations can be obtained by different approaches, being the one in the context of GR the most acclaimed. However, other approaches based on Newtonian considerations are also well-known to obtain the Friedmann equations. Although,, there exist novel approaches relying in the spirit of Jacobson's derivation of Einstein field equation (section 5.1), in which one is able to derive Friedmann equations of a Friedmann-Robertson-Walker (FRW) universe with any spatial curvature by applying the Clausius relation to apparent horizon of the FRW universe, where the entropy of the horizon is assumed to be proportional to its area. Furthermore, if one considers deviations of the area law for the entropy, then one should expect to obtain a modified Friedmann equation whose modifications are traced to the modifications of the entropy-area relationship. These ideas will be further detailed in the upcoming sections.

6.2.1 The Friedmann Equations: The Standard Cosmological Model

Let us briefly review how the well-known Friedmann equation arises from GR by solving Einstein's field equations when considering the Universe at large scales as a perfect, homogeneous and isotropic fluid described by the FRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right), \tag{6.11}$$

where $a(t)$ is the scale factor and κ is the spatial curvature. Let us consider the energy-momentum tensor for a perfect fluid

$$T_{\mu\nu} = p g_{\mu\nu} + (p+\rho) v_\mu v_\nu, \tag{6.12}$$

where ρ and p are the energy density and pressure, respectively, as measured by an observer which is at rest with respect to the fluid. In the so called comoving reference system, the energy-momentum tensor takes the diagonal form

$$(T_{\mu\nu}) = \mathrm{diag}\,(\rho, p, p, p)\,. \tag{6.13}$$

On the other hand, for the FRW metric, one finds the following non-zero components of the Ricci tensor

$$\begin{aligned} R_{00} &= \frac{3\ddot{a}}{a}, \\ R_{ij} &= 6\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}\right)\delta_{ij} \quad i,j = 1,2,3. \end{aligned}, \tag{6.14}$$

and the Ricci scalar results in

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \tag{6.15}$$

Putting all together, the Einstein field equation (2.54) gives the following differential equation for the scale factor $a(t)$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{6.16}$$

and this is the Friedmann equation. From the Einstein field equation one also obtains a second equation for $a(t)$, although, it is not an independent equation as one can obtain it by taking the time derivative of the Friedmann equation. Moreover, local conservation of energy $\nabla T = 0$ gives the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{6.17}$$

We can define the Hubble parameter as

$$H(t) = \frac{\dot{a}}{a} \tag{6.18}$$

so the friedmann equation is now written as

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}. \tag{6.19}$$

In order to solve the Friedmann equation, it is clear that one needs to provide an equation of state $\rho = \rho(p)$. Let us consider the barotropic equation of state for the perfect fluid,

$$\rho = \rho_0 a^{-3(1+\omega)} \tag{6.20}$$

where $\omega = 0$ stands for dust and $\omega = 1/3$ stands for radiation. The solution to the Friedmann equation (6.16) with $\kappa = 0$ and $\Lambda = 0$ is

$$a(t) \propto t^{\frac{2}{3(1+\omega)}}. \tag{6.21}$$

On the other hand, if we consider the Friedmann equation with only cosmological constant Λ ($\rho = 0$) we obtain the solution

$$a(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) \tag{6.22}$$

Notice the Universe grows exponentially fast in this case, which is what we know about the evolution of the Universe in current times. The cosmological constant solution can also be obtained by setting $\omega = -1$ in the equation of state (6.20), obtaining a constant energy density and an exponential behavior of the scale factor.

Of course we can consider mixtures of different components or different equations of state (REf) but this will nor be of interest in the present work.

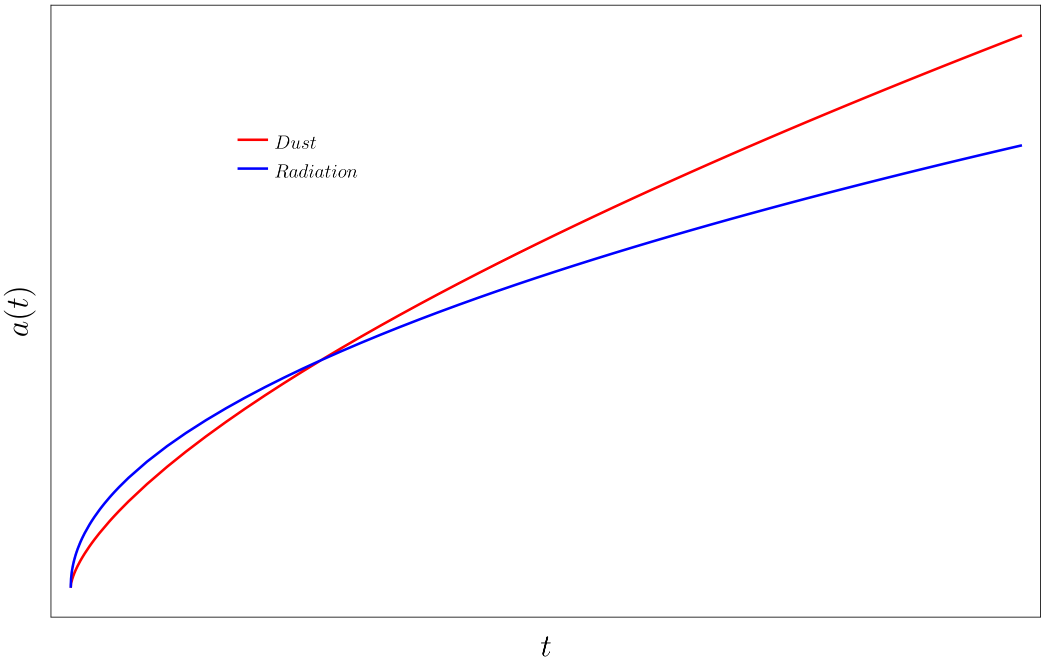

Figure 6.1: Solutions to the Friedmann equation for dust $a(t) \sim t^{2/3}$ and radiation $a(t) \sim t^{1/2}$

6.2.2 Derivation of the Friedmann Equation From Thermodynamical Considerations

Let us first review how the Friedmann equation arises by means of entropic considerations, For this purpose, we will follow the procedure explained in[26]. In this approach the evolution will be dictated by the fundamental thermodynamical law $\delta Q = TdS$ (Clausius Relation) applied on the so called apparent horizon of a FRW universe. The key idea is to demand that this relation holds on every point of the horizon, with δQ and T interpreted as the energy flux and Unruh temperature at the horizon. As for the entropy, one associates an entropy-area relationship to the horizon, first, by considering the usual $S \sim A$, where A is the horizon area.

Let us proceed with the derivation of the Friedmann equation. Let us consider the metric of a FRW universe

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right). \tag{6.23}$$

The apparent horizon is determined by the relation

$$h^{ab}\partial_a \tilde{r} \partial_b \tilde{r} = 0, \tag{6.24}$$

where h_{ab} is extracted by writing the metric as $ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega^2$. With this in mind, we can easily see that the radius of the apparent horizon is given by

$$\tilde{r}_A^2 = \left(H^2 + \kappa/a^2\right)^{-1}, \tag{6.25}$$

where $H = \dot{a}/a$ is the Hubble parameter. As usual, we consider the matter content in the form of a perfect fluid, described by the energy-momentum tensor $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$, where ρ and p are the energy density and pressure, respectively. Similarly, the energy-momentum conservation leads to the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{6.26}$$

Let us now introduce the work density W and the energy-supply vector Ψ defined as

$$W = -\frac{1}{2}T^{ab}h_{ab}, \quad \Psi_a = T_a^b \partial_b \tilde{r} + W\partial_a \tilde{r}, \tag{6.27}$$

where $\tilde{r} = a(t)r$ and T_{ab} is the projection of $T_{\mu\nu}$ in the normal direction of the 2-sphere. Using the energy-momentum tensor for a perfect fluid, the work density and the energy supply vector is

$$W = \frac{1}{2}(\rho - P), \quad \Psi_a = \left(-\frac{1}{2}(\rho + P)H\tilde{r}, \frac{1}{2}(\rho + P)a\right). \tag{6.28}$$

The amount of energy δQ crossing the apparent horizon during the time interval dt is given by[24]

$$\delta Q = -A\Psi = A(\rho + P)H\tilde{r}_A dt, \tag{6.29}$$

and $A = 4\pi\tilde{r}_A^2$ is the area of the apparent horizon. In order to proceed, we make the following assumptions: One is that the apparent horizon has am entropy proportional to its area, that is, its entropy is given by the Bekenstein-Hawking entropy, written as

$$S = \frac{A}{4G}, \tag{6.30}$$

the other one, is that the apparent horizon has a temperature

$$T = \frac{1}{2\pi\tilde{r}_A}. \tag{6.31}$$

With these considerations, the Clausius relation $\delta Q = TdS$ gives

$$4\pi\tilde{r}_A^2(\rho + P)H\tilde{r}_A dt = \frac{1}{8\pi G}\frac{1}{\tilde{r}_A}dA, \tag{6.32}$$

For dA, from (6.25) we obtain

$$dA = -8\pi\tilde{r}_A^4 H\left(\dot{H} - \frac{\kappa}{a^2}\right)dt. \tag{6.33}$$

Then, equation (6.32) becomes

$$-4\pi(\rho + P)G = \left(\dot{H} - \frac{\kappa}{a^2}\right). \tag{6.34}$$

If we use the continuity equation, together with the identity $2H(\dot{H} - \kappa/a^2)dt = d[H^2 + \kappa/a^2]$ we obtain

$$\frac{8\pi G}{3}\dot{\rho} = \frac{d}{dt}\left(H^2 + \frac{\kappa}{a^2}\right), \tag{6.35}$$

finally after integrating the last equation we arrive to the well-known Friedmann equation of FRW universe

$$\frac{8\pi G}{3}\rho = H^2 + \frac{\kappa}{a^2}, \tag{6.36}$$

which is, of course, the form of Einstein Field Equations in the metric (6.23). Thus, we have derived the Friedmann equation of a FRW universe with any spatial curvature by applying the Clausius relation to apparent horizon of the FRW universe and assuming that the apparent horizon has an entropy satisfying the area formula like black hole horizon and a temperature given by (6.31). In the following section, we will attempt to obtain a modified version of the Friedmann equation (6.36) by assuming that the horizon has, not the usual entropy (6.30), but a modified version of this entropy-area relationship, identified with the modified entropy-area relationship used in the previous section.

6.2.3 Obtaining a Modified Friedman Equation From a Modified Entropy-Area Relationship

In this section we will be using an entropy area relationship that has a term linear on the area A (this is the usual term for the GR formulation), but we also add a term proportional to $A^{3/2}$ (our volumetric contribution to the entropy). Entropy terms that scale on the volume are typically related to ordinary degrees of freedom of quantum field theories. The modified entropy-area relationship that we will use is

$$S = \frac{A}{4G} + \epsilon\left(\frac{A}{4G}\right)^{3/2}. \tag{6.37}$$

Where ϵ is a free parameter. Therefore, we expect modifications to the Friedmann equation arise from the modification added to the entropy-area relationship. The procedure to obtain the modified Friedmann equation is the same as the used for the usual Friedmann equation in the previous subsection, however, in this case, the entropy-area relationship will be given by equation (6.37).

As before, we consider the Clausius relation on the apparent horizon, assuming the horizon has an entropy given by (6.37). This gives

$$4\pi\tilde{r}_A^2(\rho+P)H\tilde{r}_A dt = \frac{1}{8\pi G}\frac{1}{\tilde{r}_A}\left[1+\frac{3\epsilon}{\sqrt{2G}}A^{1/2}\right]dA, \tag{6.38}$$

using again the expression

$$dA = -8\pi\tilde{r}_A^4 H\left(\dot{H}-\frac{\kappa}{a^2}\right)dt, \tag{6.39}$$

from Eq.(6.38) we obtain

$$-4\pi(\rho+P)G = \left[1+\frac{3\epsilon}{4\sqrt{G}}A^{1/2}\right]\left(\dot{H}-\frac{\kappa}{a^2}\right).$$

If we use the continuity equation $\dot{\rho}+3H(\rho+P)=0$, together with $2H(\dot{H}-\kappa/a^2)dt = d[H^2+\kappa/a^2]$ we get

$$\frac{8\pi G}{3}\dot{\rho} = \left[1+\frac{3}{2}\sqrt{\frac{\pi}{G}}\epsilon\left(H^2+\frac{\kappa}{a^2}\right)^{-1/2}\right]\frac{d}{dt}\left(H^2+\frac{\kappa}{a^2}\right), \tag{6.40}$$

finally after integrating Eq.(6.40) we arrive to the modified Friedmann equation

$$\frac{8\pi G}{3}\rho = \left(H^2 + \frac{\kappa}{a^2}\right) + \sqrt{\frac{9\pi}{G}}\epsilon\left(H^2 + \frac{\kappa}{a^2}\right)^{1/2}, \tag{6.41}$$

the integrating constant can be absorbed in ρ and could be interpreted as a cosmological constant. However as we are interested in studying the contributions from the volumetric term, we will set it to zero. If we set the constant to zero when using the usual entropy area relationship we get the Friedmann equation with vanishing cosmological constant and when comparing with the equations derived from the corrected entropy-area relationship we can clearly see the effects of the volumetric correction. Solving for $H^2 + \kappa/a^2$ in Eq.(6.41) we get

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho + \frac{9\pi}{2G}\epsilon^2 \pm \frac{1}{2}\sqrt{\frac{81\pi^2}{G^2}\epsilon^4 + 96\pi^2\epsilon^2\rho}. \tag{6.42}$$

6.2.4 Solving the Modified Friedmann Equation

Before solving this equation, we will center our attention to the late time behavior. In this limit, if we take the negative sign of the square root the constant term cancels out and we do not have any terms that behave as a cosmological constant. For this reason we are not interested on this branch of the solution and will focus our attention on the solution with the positive sign.

Now if we take the positive sign for the square root in Eq.(6.42) and analyze the late time behavior, it is simple to see that we get a de Sitter solution for the scale factor. This solution is given by

$$H^2 + \frac{\kappa}{a^2} \approx \frac{9\pi}{G}\epsilon^2, \tag{6.43}$$

comparing with the Friedmann equation for the de Sitter cosmological model we can identify an effective cosmological constant Λ_{eff}, which to leading order in ϵ is given by

$$\Lambda_{eff} = \frac{27\pi}{G}\epsilon^2. \tag{6.44}$$

This is a surprising result, as we have obtained an effective cosmological constant from the modification to the entropy-area relationship. From the effective cosmological constant Λ_{eff}, we can argue that the cosmological constant originates from the modification to the entropy-area relationship.

Again, let us consider the barotropic equation of state for the perfect fluid, then $\rho = \rho_0 a^{-3(1+\omega)}$. The solution to Eq.(6.41) with $\kappa = 0$, is

$$144\sqrt{\pi}G\rho_0(\omega+1)t = 3a\sqrt{96G\rho_0 a^{3\omega+1} + \frac{81\epsilon^2 a^{6\omega+4}}{G}} - \frac{27\epsilon a^{3(\omega+1)}}{\sqrt{G}} \tag{6.45}$$

$$+ \quad \frac{32G^{3/2}\rho_0 a^{-3(1+\omega)/2}\sqrt{32G^2\rho_0 a^{3\omega+1} + 27\epsilon^2 a^{6\omega+4}}\sinh^{-1}\left(\frac{3\sqrt{\frac{3}{2}}\epsilon a^{3(1+\omega)/2)}}{4G\sqrt{\rho_0}}\right)}{\epsilon\sqrt{27\epsilon^2 a^{3\omega+3} + 32G^2\rho_0}}.$$

Unfortunately, the expression for the scale factor is given in implicit form, nonetheless we can obtain the asymptotic behavior in the limit of large t, where we obtain

$$a(t) \sim \exp\left(\sqrt{\frac{9\pi}{G}}\epsilon t\right). \tag{6.46}$$

This is the kind of behavior we expected for the scale factor in the late time limit. On the other hand, for a FRW model with cosmological constant only we know that the solution for the scale factor is

$$a(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}t\right), \tag{6.47}$$

thus, comparing both solutions we can identify

$$\sqrt{\frac{9\pi}{G}}\epsilon = \sqrt{\frac{\Lambda}{3}}, \tag{6.48}$$

or

$$\Lambda = \frac{27\pi}{G}\epsilon^2, \tag{6.49}$$

which is the same expression for Λ_{eff} found before.

In fig.(1) we compare the model with the de Sitter model. If we consider dust (thick line) or radiation (thin line) we see that the scale factor in the late time scenario corresponds to a de Sitter universe. We can also notice that for early times the behavior of the model is not an exponential scale factor, this is clear in small graph in the fig.1, where we have zoomed the plot for small t.

In order to understand the behavior before exponential growth, we will explore the small t limit of Eq.(6.45). In this limit, we get for the scale factor

$$a(t) \sim \begin{cases} t^{\frac{2}{3}}, & \text{for } \omega = 0, \\ t^{\frac{1}{2}}, & \text{for } \omega = 1/3. \end{cases} \tag{6.50}$$

Which are the same results as those of the usual Friedmann equation with a perfect fluid and a barotropic equation of state as described in subsection 6.2.1. In fig.(2) we compare the solution to our model with the solution for the usual Friedmann equation, and we can see that there is a correspondence between the two models. For small t the volumetric term of the entropy has negligible effects and therefore we have the same behavior as that of the usual FRW universe, while for late times the volumetric contribution dominates the dynamics and gives scale factor that behaves similarly to a dark energy dominated universe.

As pointed out before, we can define an effective cosmological constant Λ_{eff} in terms of the parameter ϵ, that is associated to the volumetric contribution to the entropy. Using the current observed value for the cosmological constant Λ, we fix the value of the parameter $\epsilon \sim 10^{-60}$. This gives a simple solution to the origin of Λ in the context of entropic gravity.

Finally we will like to point out that the modified entropy that gives the effective cosmological

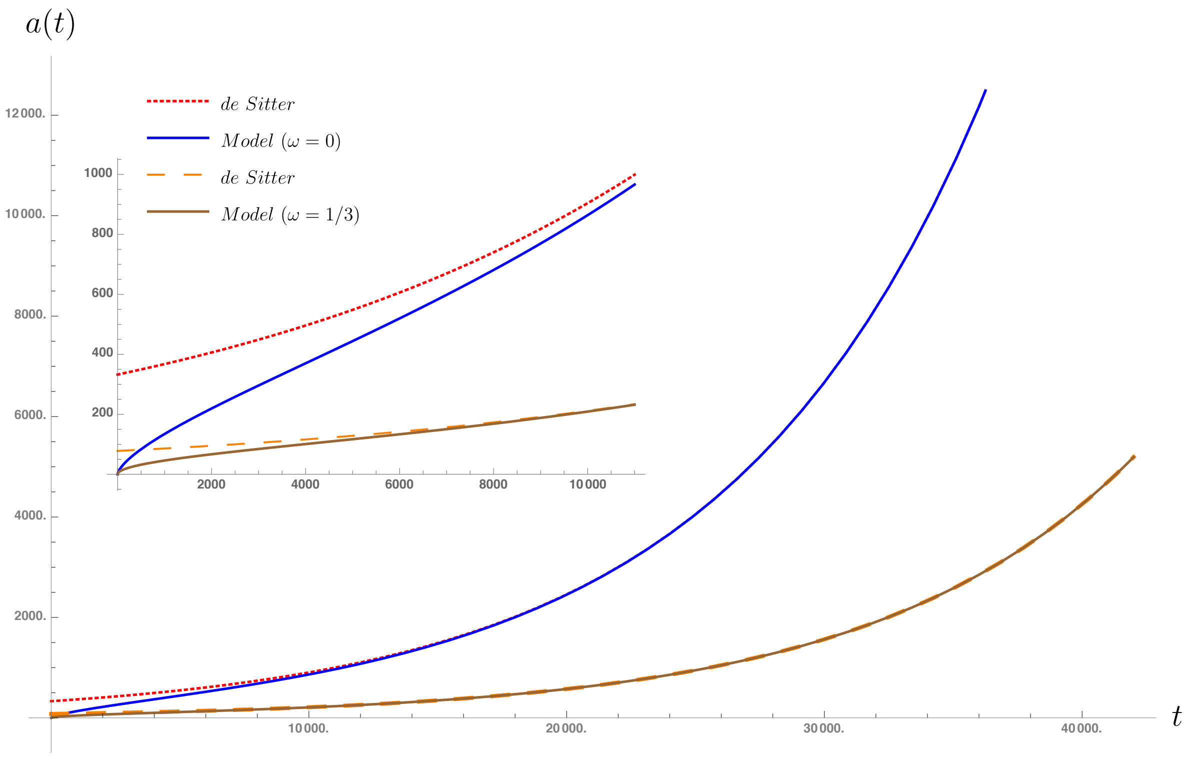

Figure 6.2: In this plot we compare the scale factor of this model with that of a de Sitter Universe. The thick line corresponds to our model with dust, Eq.(6.45) with $\omega = 0$. The dotted line corresponds to a de Sitter universe and one can clearly see that for large t the behavior is the same. For radiation, $\omega = 1/3$, the thin line corresponds to our model and the dashed line for a de Sitter Universe, as before we see that the two curves are exponential for large t. In the two cases($\omega = 0$ and $\omega = 1/3$) our model differs to the de Sitter case for small t, this can be seen in the box. For all plots we took $\rho_0 = \frac{3}{8\pi G}$ and $\epsilon = 10^{-4}$.

constant Λ_{eff}, is the same modified entropy that gives the correction to the Newtonian gravitational force that accounts for the anomalous galactic rotation curves. If we estimate the value of ϵ from the result in [14], using the mass of a typical galaxy (i.e. our galaxy) we get $\epsilon \sim 10^{-58}$, this value is consistent with the estimated value we have obtained from Eq.(6.44). Currently we can only say that it is an interesting coincidence, but it might point to a deeper relationship, this is currently under research and will be reported elsewhere.

It is worth to be added that further analysis can be made on the modified Friedmann equation obtained in this work. For instance, by defining a parameter associated with the parameter of our model ϵ as

$$\Omega_\epsilon = \frac{1}{2H^2}\sqrt{\frac{9\pi}{G}}\epsilon, \tag{6.51}$$

together with the other parameters associated to matter and spatial curvature

$$\Omega_m = \frac{8\pi G}{3H^2}\rho, \quad \Omega_\kappa = -\frac{\kappa}{a^2H^2}, \tag{6.52}$$

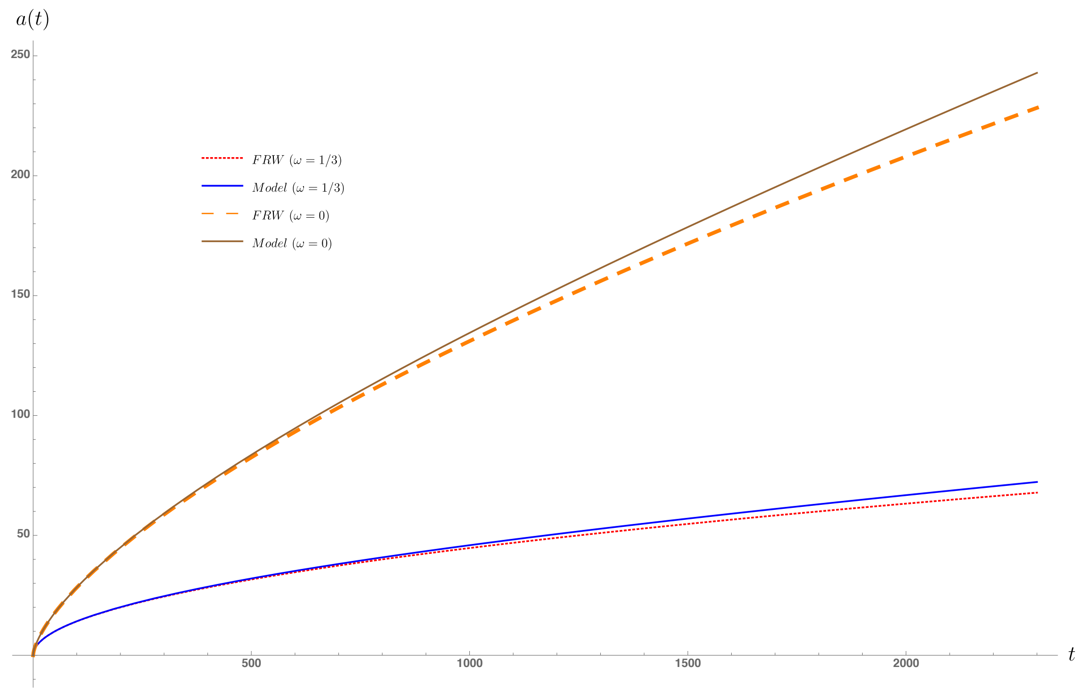

Figure 6.3: This plot is a comparison between our model and the standard FRW cosmology with a barotropic fluid and with the same matter content. The thick line corresponds to our model with dust (Eq.(6.45) with $\omega = 0$), the dashed line is for the FRW model with dust. One can clearly see that for early time the behavior is the same. For radiation ($\omega = 1/3$), the thin line corresponds to our model and the dotted solid line for the FRW model with radiation, as before the to models have the same behavior for small t. For all plots we took $\rho_0 = \frac{3}{8\pi G}$ and $\epsilon = 10^{-4}$.

equation (6.41) can be written as

$$1 = \Omega_m + \Omega_\kappa + \Omega_\epsilon \pm \sqrt{\Omega_\epsilon^2 + 2\Omega_\epsilon \Omega_m}. \tag{6.53}$$

The latest equation is to be recognized as the modified constriction for the densities parameters. Of course, by choosing the plus sign in equation (6.42) motivates to choose it in this equation as well.

Now, for example if we consider dust, that is, $\rho = \rho_0 a^{-3}$, then we have

$$\Omega_0 \equiv \Omega_{0m} = \frac{8\pi G}{3H_0^2}\rho_0, \quad \Omega_{0\kappa} = -\kappa/H_0^2, \quad \Omega_\epsilon = \Omega_{0\epsilon}, \tag{6.54}$$

where H_0 is the Hubble constant (current value of the Hubble parameter H), and equation (6.42) becomes

$$H^2 = H_0^2 \left(\Omega_0 a^{-3} + \Omega_{0\kappa} a^{-2} + \Omega_{0\epsilon} + \sqrt{\Omega_{0\epsilon}^2 + 2\Omega_{0\epsilon}\Omega_{0m} a^{-3}} \right). \tag{6.55}$$

In cosmology it is convenient to express the quantities in terms of the redshift z, which is related to the scale factor a as

$$a = \frac{1}{1+z}, \tag{6.56}$$

with this, we can write

$$H^2(z) = H_0^2\left[\Omega_0(1+z)^3 + \Omega_{0\kappa}(1+z)^2 + \Omega_{0\epsilon} + \sqrt{\Omega_{0\epsilon}^2 + 2\Omega_{0\epsilon}\Omega_{0m}(1+z)^3}\right]. \tag{6.57}$$

Finally we obtain the equation that dictates the evolution of the matter parameter Ω_m

$$\Omega_m(z) = \Omega_0\frac{(1+z)^3}{\Omega_0(1+z)^3 + \Omega_{0\kappa}(1+z)^2 + \Omega_{0\epsilon} + \sqrt{\Omega_{0\epsilon}^2 + 2\Omega_{0\epsilon}\Omega_{0m}(1+z)^3}}. \tag{6.58}$$

A deeper cosmological analysis can be made by formally solving the system of equations (6.53) and (6.58) in order to obtain the evolution of the matter density parameter predicted by this model. This is under consideration and will be worked out in the future.

Moreover, we can consider the more general entropy-area relationship (see appendix A)

$$S = \frac{A}{4G} + \Gamma(\epsilon)\left(\frac{A}{4G}\right)^{1/2} + \epsilon\left(\frac{A}{4G}\right)^{3/2}. \tag{6.59}$$

If we follow the procedure to obtain the modified Friedmann equation corresponding to this entropy, just as we previously did in this section, we obtain the modified Friedmann equation

$$\frac{8\pi G}{3}\rho = H^2 + \frac{\kappa}{a^2} + \alpha(\epsilon)\left(H^2 + \frac{\kappa}{a^2}\right)^{3/2} + \beta(\epsilon)\left(H^2 + \frac{\kappa}{a^2}\right)^{1/2}, \tag{6.60}$$

where $\alpha(\epsilon), \beta(\epsilon)$ are some functions of the parameter ϵ. This more general modified Friedmann equation has not been solved yet, and of course, solving it would let us inquire about the repercussions of the new term $A^{1/2}$ onto the evolution of the universe, and perhaps this could be related with the predictions of some other cosmological model.

Chapter 7

Conclusions and Final Remarks

The complexities of defining a quantum theory of gravity has led to consider gravity as an emergent phenomena. This opens a new paradigm for the understanding of the origin of the gravitational interaction.

In the work described in section 4 we have considered a generalized Stelle-West formulation by means of introducing a pseudoprojector Π acting on $\mathfrak{so}(4,1)$ Lie algebra valued field strength, this projector has the shape of a complex (anti)self-dual projector in a Plebański like formulation. Likewise, as the original Stelle-West formulation the vector field v hidden into the projector Π, plays a central role by breaking the symmetry down to $SO(3,1)$ by considering SBC, but this time, along with Einstein-Cartan theory plus the Euler class, we have also obtained the Holst modification with Immirzi parameter, the Nieh-Yan form and the Pontryagin invariant. Then, we went further by considering the action without the SBC. The equations of motion imposed by this action are not trivial, so we had to impose some constraints on the covariant derivative of v. First, we imposed a condition that let us recover back the Einstein field equations, but the equation of motion for v imposed a condition over the cosmological constant, so Λ is calculated within the theory, rather than being a free parameter that has to be measured. The second case we have considered, let us construct topological torsionless field theories, where the Minkowski part of the vector v is orthogonal to the spin connection and $\chi = 0$. Also, since the Lie algebra is a reductive algebra, and the connection is an $\mathfrak{so}(3,1)$ valued field, then the vector v tell us how to construct locally the SO(3,1)-bundle together with the frame bundle. It will be interesting to find numerical solutions for the field v, under the second condition, for topological torsionless field theories, obtaining the explicit form of the spin connection and the tetrad field. As a result, we can calculate the metric field itself that let us characterize certain resulting manifolds and their topological structures.
A complex (anti)self-dual formulation could be compelling to consider in a future work. Since we have investigated just the pure bosonic case, it will be fascinating to consider also a supersymmetric extension for $\mathcal{N} = 1$.
Also, it will be intriguing to compare the relationship between the action in eq.(4.9) and the quasitopological principle proposed by Alexander et. al. in [34], for their θ term and the non-constant cosmological constant Λ.

A higher dimensional internal symmetry group based on MM approach for pure connection formulations of gravity for real fields, inspired in [51] may be considered in order to obtain a family of torsionless conformally flat Einstein manifolds, this work is in progress and will be reported elsewhere.

By assuming an entropic origin of Newtonian gravity we can study new effects by considering modifications to the Bekenstein-Hawking entropy. In this work, we used the supersymmetric minisuperspace approach for the Schwarzschild black hole to construct a supersymmetric generalization and from the Feynman-Hibbs approach we obtained the entropy-area relationship. It is worth mentioning, that except for the logarithmic term that has a quantum origin, the remaining terms are related to the supersymmetric modification. Assuming an entropic origin to gravity, we constructed a generalized Newtonian force, the modified gravitational potential and the effective matter density. When considering the case of circular orbit for very large radius, this modified theory of gravity can account for the anomalous galaxy rotation curves. Therefore one can conjecture that if gravity is emergent, supersymmetry can substitute (under some conditions) the need for dark matter to explain the rotation curves. Although this results are encouraging, further exploration is needed to establish this model as a replacement to dark matter.

In this work we set our interest on the origin of the cosmological constant. Using the new entropy-area relationship, we derive a modified Friedmann equation. Focusing our interest in the late-time limit, we find out that the asymptotic behavior is that of a de Sitter cosmology. Then one can define an effective cosmological constant Λ_{eff} in terms of the model parameter ϵ. This gives a simple solution to the origin of Λ in the context of entropic gravity. Also it is of great interest for us to test our models with observational data, this would allow us to obtain a better bound for the parameter ϵ. Finally It remains to be seen why the obtained values for ϵ are of similar order for two totally different phenomena: the anomaly of rotation curves and the cosmological scenario. Is it possible that what we know as dark energy and dark matter are somehow connected with each other? For example, it would be perhaps daring but interesting to consider the following: Since from the modified Friedmann equation (6.41) we found an effective cosmological constant in terms of the parameter ϵ, we can invert this relationship and express ϵ in terms of Λ. On the other hand, we know that the rotation velocity (6.5) depends on ϵ, so it would be possible to express it in terms of Λ, in this way, we could claim that dark energy could influence the phenomenon of the anomaly of rotation curves in galaxies.

Appendix A

Derivation of the Modified Entropy-Area Relationship

A.1 The supersymmetric WDW equation for the Schwarzschild Metric

The Schwarzschild black hole is described by the metric

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \\ & + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned} \tag{A.1}$$

for the case $r < 2m$ the g_{rr} and the g_{tt} components of the metric change in sign and ∂_t becomes a space-like vector, hence if we perform the transformation $t \leftrightarrow r$, we find

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2m}{t}\right)^{-1} dt^2 + \left(1 - \frac{2m}{t}\right) dr^2 \\ & + t^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \end{aligned} \tag{A.2}$$

When compared with the Misner parametrization of the KS metric

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\xi} dr^2 + e^{-2\sqrt{3}(\xi+\Omega)} \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{A.3}$$

we identify

$$N^2 = \left(\frac{2m}{t} - 1\right)^{-1}, e^{2\sqrt{3}\xi} = \frac{2m}{t} - 1, e^{-2\sqrt{3}(\xi+\Omega)} = t^2, \tag{A.4}$$

this establishes the diffeomorphism with the Kantowski-Sachs metric. Using this result in the Einstein-Hilbert action and performing an integration over the spatial coordinates, we get an effective Lagrangian from which it is straight forward to obtain the Hamiltonian constraint

$$H = p_\xi^2 - p_\Omega^2 - 48 e^{-2\sqrt{3}\Omega}. \tag{A.5}$$

From standard canonical quantization, with the usual identifications for the canonical momenta $p_\xi = -i\frac{\partial}{\partial \xi}$, $p_\Omega = -i\frac{\partial}{\partial \Omega}$, the WDW equation derived from the Hamiltonian constraint is

$$\left(-\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \xi^2} + 48e^{-2\sqrt{3}\Omega}\right)\Psi(\Omega,\xi) = 0, \tag{A.6}$$

Using the plane wave solution for the variable ξ, the WDW equation takes the form

$$\left(-\frac{d^2}{d\Omega^2} + 48e^{-2\sqrt{3}\Omega}\right)\chi(\Omega) = 3\nu^2\chi(\Omega). \tag{A.7}$$

We now proceed to obtain the supersymmetric version of the WDW equation Eq.(A.6), for this purpose we will follow [52]. For homogeneous models the Hamiltonian H_0 can be written as

$$2H_0 = \mathcal{G}^{\mu\nu}p_\mu p_\nu + \mathcal{U}(q^\mu), \tag{A.8}$$

where q^μ are the minisuperspace coordinates, $\mathcal{G}^{\mu\nu}$ is the minisuperspace metric and $\mathcal{U}(q^\mu)$ is the potential. Also, it is possible to find a function $\Phi(q^\nu)$ satisfying

$$\mathcal{G}^{\mu\nu}\frac{\partial \Phi}{\partial q^\mu}\frac{\partial \Phi}{\partial q^\nu} = \mathcal{U}(q^\alpha). \tag{A.9}$$

To construct the supersymmetric Hamiltonian first we need the supercharges

$$\mathcal{Q} = \psi^\mu\left(p_\mu + i\frac{\partial \Phi}{\partial q^\mu}\right), \quad \bar{\mathcal{Q}} = \bar{\psi}^\mu\left(p_\mu - i\frac{\partial \Phi}{\partial q^\mu}\right), \tag{A.10}$$

where $\bar{\psi}^\mu$, and ψ^ν are Grassmann variables and satisfy the algebra

$$\{\bar{\psi}^\mu, \bar{\psi}^\nu\} = \{\psi^\mu, \psi^\nu\} = 0, \quad \{\bar{\psi}^\mu, \psi^\nu\} = \mathcal{G}^{\mu\nu}. \tag{A.11}$$

The supersymmetric Hamiltonian is obtained from the algebra of the supercharges $2H_S = \{\mathcal{Q}, \bar{\mathcal{Q}}\}$, this gives

$$2H_S = \mathcal{G}^{\mu\nu}p_\mu p_\nu + \mathcal{U}(q^\mu) + \frac{\partial^2 \Phi}{\partial q^\mu \partial q^\nu}\left[\bar{\psi}^\mu, \psi^\nu\right]. \tag{A.12}$$

This supersymmetric Hamiltonian is the supersymmetric generalization of Eq.(A.8), it is the sum of the "bosonic" Hamiltonian and the contribution $\frac{\partial^2 \Phi}{\partial q^\mu \partial q^\nu}$. This Hamiltonian is fully determined once we adopt a suitable representation of the Grassmann variables

$$\begin{aligned}
\psi^\xi &= \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&-1&0 \end{pmatrix}, \quad \bar{\psi}^\xi = \begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0 \end{pmatrix},\\
\psi^\Omega &= \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}, \quad \bar{\psi}^\Omega = \begin{pmatrix} 0&0&-1&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}.
\end{aligned} \tag{A.13}$$

From Eq.(A.6) we identify $\mathcal{U}(\Omega,\xi) = 48e^{-2\sqrt{3}\Omega}$ and from Eq.(A.9) we obtain a differential equation for Φ

$$\left(\frac{\partial \Phi}{\partial \Omega}\right)^2 - \left(\frac{\partial \Phi}{\partial \xi}\right)^2 = 48e^{-2\sqrt{3}\Omega}. \tag{A.14}$$

The solution for Φ is given by

$$\Phi = -4\left[\sqrt{e^{-2\sqrt{3}\Omega}+\epsilon^{-2/3}} - \epsilon^{-1/3}\text{arcsinh}\left(\epsilon^{-1/3}e^{\sqrt{3}\Omega}\right)\right] + 4\sqrt{3}\epsilon^{-1/3}\xi, \tag{A.15}$$

with $\epsilon =$ constant. Since $\left[\bar{\psi}^{\Omega}, \psi^{\Omega}\right] = \text{diag}(-1,-1,1,1)$, the supersymmetric Hamiltonian will have two independent components which only differ in the sign of the modified potential. Now the proposed WDW equation for the supersymmetric quantum Schwarzschild black hole is

$$\left[-\frac{\partial^2}{\partial\Omega^2} + \frac{\partial^2}{\partial\xi^2} + 12\left(4 \pm \frac{1}{\sqrt{e^{-2\sqrt{3}\Omega}+\epsilon^{-2/3}}}\right)e^{-2\sqrt{3}\Omega}\right]\Psi^S_{\pm}(\Omega,\xi) = 0. \tag{A.16}$$

The wave function has four components, although only two are linearly independent, also the contributions of supersymmetry are encoded in the modified potential. Finally, it is worth mentioning that Eq.(A.6) is recovered from Eq.(A.16), by taking the limit $\epsilon \to 0$.

Calculation of the Modified Entropy-Area relationship

Let us start by reviewing the calculation of the entropy for the Schwarzschild black hole, using the Feynman-Hibbs procedure. This approach was originally applied to the Schwarzschild black hole [41] and subsequently used for different black hole models [53–55]. In the limit of small Ω and taking $x = l_P(\sqrt{6}\Omega - 1/\sqrt{2})$, Eq.(A.7) can be written as

$$\left(-\frac{1}{2}l_P^2 E_P\frac{d^2}{dx^2} + 4\frac{E_P}{l_P^2}x^2\right)\chi(x) = E_P\left(\frac{\nu^2}{4}-2\right)\chi(x), \tag{A.17}$$

we can see that Eq.(A.17) is the usual quantum harmonic oscillator if we identify $\hbar\omega = \sqrt{\frac{3}{2\pi}}E_p$ and $\frac{\hbar^2}{m} = l_P^2 E_P$.

To compute the "corrected" partition function of the black hole we apply the Feynman-Hibbs procedure. This approach is based on exploiting the similarities of the expression of the density matrix and the kernel of Feynman's path integral approach to quantum mechanics. By doing a Wick rotation $t \to i\beta$, we get the Boltzmann factor and the kernel is transformed to the density matrix. The kernel is calculated along the paths that go from x_1 to x_2, if we consider small Δt (small β). Then, when calculating the partition function, only the paths that stay near x_1 have a non-negligible contribution (the exponential in the expression for the density matrix gives a negligible contribution to the sum from the other paths). Therefore, the potential to a first-order approximation can be written as $V(x) \approx V(x_1)$ for all the contributing paths. In this approximation, we can formally establish a map from the path integral formulation of quantum mechanics to the classical canonical partition function. To introduce quantum effects, we must incorporate the changes to the potential along the path; in particular, we are interested in the first-order effects. For this, we start by doing a Taylor expansion around the mean position $\tilde{x}$ along any path. Calculating the kernel with $\tilde{x}$ and doing the Wick rotation, we get the modified partition function. This partition function is calculated in a classical manner, but with the corrected potential, which is a mean value of the potential $V(x)$ averaged over points near $\tilde{x}$ with a Gaussian distribution. Therefore, to calculate the partition function with quantum corrections [41, 53, 54], we use the corrected potential. According to this procedure, the corrected partition function is

$$Z = \sqrt{\frac{m}{2\pi\beta\hbar^2}} \int_{-\infty}^{\infty} e^{-\beta U(\tilde{x})} d\tilde{x}, \tag{A.18}$$

where $\beta = 1/k_B T$ and $U(\tilde{x})$ is the corrected potential given by

$$U(\tilde{x}) = \sqrt{\frac{12m}{2\pi\beta\hbar^2}} \int_{-\infty}^{\infty} V(\tilde{x}+y) e^{-6y^2 m/\beta\hbar^2} dy. \tag{A.19}$$

Now we substitute the potential of the WDW equation in Eq.(A.18) and Eq.(A.19), this gives the corrected partition function

$$Z = \sqrt{\frac{2\pi}{3}} \frac{1}{\beta E_P} e^{-\beta^2 E_P^2/16\pi}. \tag{A.20}$$

From the partition function it is straightforward to calculate the temperature and entropy for the Schwarzschild black hole. We begin with the internal energy

$$E = -\frac{\partial}{\partial\beta} \ln Z, \tag{A.21}$$

which gives a relation between the black hole corrected temperature β and its mass M. In terms of Hawking temperature $\beta_H = \frac{8\pi M c^2}{E_P^2}$, the corrected temperature of the black hole is

$$\beta = \beta_H \left(1 - \frac{1}{\beta_H} \frac{1}{Mc^2}\right), \tag{A.22}$$

where we can observe an extra contribution to the temperature proportional to β_H^{-1}.

To calculate the entropy we use

$$\frac{S}{k_B} = \ln Z + \beta E, \tag{A.23}$$

by relating the Bekenstein-Hawking entropy to the Hawking temperature as $Mc^2\beta_H = 2\frac{S_{BH}}{k_B}$ we get entropy

$$\frac{S}{k_B} = \frac{S_{BH}}{k_B} - \frac{1}{2} \ln \frac{S_{BH}}{k_B} + \mathcal{O}(S_{BH}^{-1}). \tag{A.24}$$

This result has the interesting feature that the logarithmic correction agrees with the one obtained in string theory as well as in loop quantum gravity [42–44].

Now we apply this method to obtain the temperature and entropy of the supersymmetric Schwarzschild black hole. Since the potential in Eq.(A.16) depends only on the Ω coordinate, we use the plane wave solution for ξ. Following the same procedure and similar approximations as in the original case, the equation takes the form

$$\left[-\frac{1}{2} l_P^2 E_P \frac{d^2}{dx^2} + 4\frac{E_P}{l_P^2} x^2 \pm \frac{1}{2} \frac{E_P}{l_P^4} \epsilon x^4\right] \chi_\pm^S(x) = \eta^2 \chi_\pm^S(x). \tag{A.25}$$

From now on we will take the $(-)$ case, the $(+)$ case follows straightforwardly by transforming $\epsilon \to -\epsilon$.

Now we apply the Feynman-Hibbs procedure for the potential of the form $V(x) = \frac{m\omega^2}{2}x^2 + \lambda x^4$. Following the bosonic case a straightforward calculation gives the corrected partition function for the supersymmetric model

$$Z_S = \sqrt{\frac{2\pi}{3}}\frac{1}{\beta E_P}\exp\left(-\frac{\beta^2 E_P^2}{16\pi} - \frac{\beta^3 E_P^3 \epsilon}{96}\right)\left(1 + \frac{\pi \beta E_P \epsilon}{3}\right)^{-1/2}.$$

As before, this partition function reduces to the original one in the limit $\epsilon \to 0$. For temperature we proceed as before, using the partition function Eq.(A.26) the internal energy is

$$\frac{1}{\beta} + \frac{E_P^2\beta}{8\pi} + \frac{E_P^3\beta^2}{32}\epsilon + \frac{\pi E_P}{6}\epsilon - \frac{\pi^2 E_P^2 \beta}{18}\epsilon^2 = Mc^2, \tag{A.26}$$

solving for β in terms of the Hawking temperature gives

$$\beta = \beta_H\left[1 - \frac{1}{\beta_H Mc^2} + f(\epsilon)\frac{1}{(\beta_H Mc^2)^{1/2}}\right], \tag{A.27}$$

where $f(\epsilon) = \frac{2}{3}\epsilon^3 - \epsilon$, and for convenience, the parameter ϵ has been redefined by $\epsilon \to \frac{2^{1/2}\pi^{3/2}}{3}\epsilon$. We see the supersymmetric contribution to the temperature is proportional to $\beta_H^{-1/2}$, also in the limit $\epsilon \to 0$ we recover Eq. (A.22).

From the partition function Eq.(A.26) and the temperature, the entropy for the supersymmetric model is given by

$$\frac{S}{k_B} = -\frac{1}{2}\ln\frac{3E_P^2\beta^2}{2\pi} + \frac{E_P^2\beta^2}{16\pi} + \frac{E_P^3\beta^3}{48}\epsilon - \frac{\pi^2 E_P^2\beta^2}{18}\epsilon^2 + 1. \tag{A.28}$$

Solving in terms of the Bekenstein-Hawking entropy $S_{BH}/k = A/4l_P^2$ we arrive to the entropy-area relationship

$$\frac{S[A]}{k_B} = \frac{[1+\Delta(\epsilon)]}{4l_P^2}A - \frac{1}{2}\ln\frac{A}{4l_P^2} + \Gamma(\epsilon)\left(\frac{A}{4l_P^2}\right)^{1/2} + \epsilon\left(\frac{A}{2l_P^2}\right)^{3/2}, \tag{A.29}$$

where $\Gamma(\epsilon) = 2^{1/2}f(\epsilon) - 3\cdot 2^{1/2}\epsilon + 3\cdot 2^{1/2}\epsilon f^2(\epsilon) - 2^{5/2}\epsilon^2 f(\epsilon)$ and $\Delta(\epsilon) = 6\epsilon f(\epsilon) - 4\epsilon^2$.

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León, Gto, 27 de Noviembre de 2020

Asunto: Carta liberación Alberto Isaac Díaz Saldaña

Dr. David Delepinde
Director de la DCI

Como miembro del Sinodo del estudiante de Doctorado en Física **Alberto Isaac Díaz Saldaña**, me permito comentar que la investigación de trabajo de tesis denominado ***Alternative Formulation of Gravity,*** cuya revisión realice, esta dentro de los parámetros de una tesis doctoral. El C. Díaz Saldaña tiene dos trabajos publicados sobre su tema de investigación, y uno más enviado que esperan sea publicado dada la importancia del tema desarrollado.

Por lo cual considero que los temas tratados durante su periodo doctoral es más que suficiente para obtener un grado de Doctor en Física por parte de la Universidad de Guanajuato, por lo que autorizo a seguir el proceso para poder hacer su disertación pública de este trabajo lo antes posible.

"LA VERDAD OS HARA LIBRES"

Dr. José Socorro García Díaz
Tutor-Sinodal

León, Guanajuato, 03 de Diciembre de 2020.

Dr. David Y. Delepine
Director
División de Ciencias e Ingenierías,CLE
P R E S E N T E

Por este conducto me permito comunicarle que, en relación a la tesis: **"Alternative formulations of gravity"**, que presenta el M.F. **Alberto Isaac Díaz Saldaña** para obtener el grado de Doctor en Física, he leído detenidamente el documento, aportando sugerencias, correcciones, y discutido ampliamente su contenido con el alumno.

Después de lo anterior expreso mi conformidad con su contenido considerando el trabajo del nivel adecuado como trabajo doctoral, no teniendo inconveniente en que el mencionado trabajo sea defendido por el interesado cuando a él convenga.

Sin otro particular, aprovecho la presente para enviarle cordiales saludos.

ATENTAMENTE
"LA VERDAD OS HARÁ LIBRES"

José Torres A.

Dr. José Torres Arenas

León Gto. A 04 de diciembre de 2020

Dr. David Delepine
Director de la División de Ciencias e Ingenierías
Campus León
Universidad de Guanajuato
PRESENTE

Estimado Dr. Delepine,

En mi calidad miembro del comité de sinodales del alumno M.F. Alberto Isaac Díaz Saldaña, por este medio informo a usted que he revisado su tesis doctoral titulada *"Alternative Formulations of Gravity"* que desarrolló Isaac con el fin de obtener el grado de Doctor en Física.

El trabajo de Isaac posee el contenido y la relevancia necesaria como trabajo de investigación. Considero que su trabajo de tesis está listo para ser defendido públicamente.

Sin más por el momento, me despido con un cordial saludo.

ATENTAMENTE
"LA VERDAD OS HARÁ LIBRES"

Dr. José Luis López Picón
Departamento de Física, DCI.

Zacatecas, Zac, a 3 de diciembre de 2020

Dr. David Delepine
Director de la División de Ciencias e Ingenierías, campus León
Universidad de Guanajuato
PRESENTE

Estimado Dr. Delepine, por este conducto hago de su conocimiento que he revisado el trabajo de tesis de doctorado titulado *"Alternative theories of gravity"* que presenta el **M. en F. Alberto Isaac Díaz Saldaña**. Este trabajo ha sido revisado extensamente en sus contenidos teóricos y prácticos, así como su presentación (incluyendo redacción y ortografía), los cuales resultan ser adecuados. Con base a esto considero que dicho trabajo manifiesta los elementos necesarios y suficientes para ser presentados como tesis del Programa Académico de Doctorado en Física y con ello aspirar a obtener el grado de Doctor en Física.

Esperando que la presente sirva para proceder con los trámites administrativos conducentes y sin otro particular por el momento, me despido de usted enviándole un cordial saludo.

Atentamente

Julio C López D.

Dr. Julio César López Domínguez
Unidad Académica de Física
Universidad Autónoma de Zacatecas

c.c.p archivo

BUAP

Puebla, Pue., a 2 de diciembre de 2020.

Dr. David Delephine
Director de la División de Ciencias e Ingenierías
Campus León, Universidad de Guanajuato
PRESENTE

Estimado Dr. Delephine

Por medio de la presente, me permito responder a la solicitud de revisión del trabajo de tesis titulado “**Alternative Theories of Gravity**” que realizó el Maestro en Física **Alberto Isaac Díaz Saldaña** y dirigido por el Dr. Oscar Miguel Sabido Moreno y el Dr. Julio César López Domínguez, con el fin de obtener el grado de Doctor en Física que otorga la Universidad de Guanajuato.

Después de haber leído detenidamente el escrito y de discutir el tema con Isaac, considero que su trabajo de tesis tiene la profundidad, claridad, extensión y originalidad necesaria para ser presentada en un examen para obtener el grado de Doctor en Física.

Sin más por el momento y agradeciendo su valioso tiempo, me despido. Saludos cordiales.

Dr. José Eduardo Rosales Quintero
Preparatoria 2 de octubre de 1968
Benemérita Universidad Autónoma de Puebla

León Guanajuato, a; 2 de Diciembre de 2020

Dr. David Delepine
Director de la División de Ciencias e Ingenierías
Campus León, Universidad de Guanajuato
PRESENTE

Estimado Dr. Delepine:

Por este medio, me permito informarle que he leído y revisado la tesis titulada **"Alternative Theories of Gravity "** que realizó el Maestro en Física **Alberto Isaac Díaz Saldaña**, como requisito para obtener el grado de Doctor en Física.

Considero que el trabajo de doctorado realizado por Isaac es de interés para la comunidad científica del área de gravitación y por lo tanto reúne los requisitos necesarios de calidad e interés académico para que sea defendida en un examen profesional, razón por la cual extiendo mi aval para que así se proceda.

Sin más que agregar, agradezco su atención y aprovecho la ocasión para enviarle un cordial saludo.

ATENTAMENTE
"LA VERDAD OS HARÁ LIBRES"

DR. Juan Barranco Monarca
Departamento de Física
DCI, Campus León

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http://www.ifug.ugto.mx

Asunto: Revisión de Tesis de Doctorado.

León, Gto., a 1 de diciembre del 2020.

Dr. David Delepine
Director de la DCI-UG
Campus León.
Presente:

Por medio de la presente quiero responder a la solicitud de revisión del trabajo de tesis titulado ***"Alternative Formulations of Gravity"***, realizado por el M. en F. Alberto Isaac Díaz Saldaña.

Después de leer el trabajo en el que se estudian teorías alternativas de gravedad. En la primera parte de la tesis trata con la generalización de la teoría MacDowell-Mansouri introduciendo nuevos campos a la teoría. Se analizan las condiciones de rompimiento para escribir las ecuaciones de campo y encontrando que la constante cosmológica es el cociente del término de Euler entre el volumen. La segunda parte, trata con el estudio de una formulación entrópica de la gravedad. Se estudian las consecuencias a la escala galáctica como la cosmológica, concluyendo que una corrección volumétrica a la entropía de Hawking-Bekenstein puede utilizarse en el contexto de materia y energía oscura.

En base a esto concluyo que es un estudio interesante y muy relevante en la gravitación y puede dar luz a problemas fundamentales como el del sector oscuro del universo y la cuantización de la gravedad.

Por lo mencionado anteriormente creo que el trabajo de tesis ha sido concluido, es original y esta listo para defensa pública.

Sin más por el momento

Atentamente

Dr. Oscar Miguel Sabido Moreno.

C.c.p. Archivo

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