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# Numerical implementation of quantum control protocols based on "on-off" stages for a single qubit

## Implementación numérica de algoritmos de control cuántico basados en etapas de prendido y apagado para un *qubit*

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### ABSTRACT

In this paper the stability of specific quantum control protocols is analyzed through a simulation. Two-level quantum systems driven by resonant external fields are studied. The focus of attention are those protocols requiring long sequences of external pulses. It is proven that these techniques are prone to cumulative error. Inspired by the methodology of the treatment of noise in signal processing, noise effects of practical implementations of single qubit quantum algorithms are also considered. The numerical results are consistent when compared to experimental data previously reported.

### RESUMEN

Se analiza la estabilidad de algoritmos específicos de control cuántico a través de la simulación. Se estudian sistemas cuánticos de dos niveles manipulados por campos resonantes externos. Nos enfocamos en aquellos protocolos que requieren secuencias grandes de pulsos externos. Se prueba que estas técnicas son susceptibles de errores de acarreo. Inspirados por la metodología de tratamiento de señales, se consideran efectos de ruidos en implementaciones prácticas de algoritmos cuánticos para un *qubit*. Los resultados encontrados están en concordancia con datos experimentales reportados.

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**Palabras clave :**
Control cuántico; ruido; rotaciones de Rabi; compuertas cuánticas; algoritmos cuánticos.

### INTRODUCTION

The quantum world has already provided outstanding technological benefits. It has extended our life expectancy through its applications in medical science; it has also improved communication and information processing through digital electronics and integrated circuits, having a huge impact on our lifestyle. Additionally, the study of quantum systems holds great potential to revolutionize science as it is understood nowdays due to experimental and theoretical developments.

In quantum systems richness and complexity go hand in hand. Their manipulation is a highly complex matter from both the experimental and the theoretical point of view. One would expect, however, that if we have been able to engineer technology using those systems, we would be able to manipulate them at will.

Progress in quantum control theory not only allows the clarification and understanding of quantum mechanics (Fernández, 1998; Fernández & Mielnik, 1994; Fernández & Rosas-Ortiz, 1997; Harel & Akulin, 1999; Mielnik, 1986, 1974, 1977), but it also enables several novel applications such as *e.g.* particle trapping and cooling (Bushev *et al.*, 2006; Gomez, Aubin, Orozco, Sprouse, Iskrenova-Tchoukova & Safronova, 2008; Paul, 1990; Smith, Reiner, Orozco, Kuhr & Wiseman, 2002), quantum state preparation (Lamb, 1969; Van Handel, Stockton & Mabuchi, 2005), squeezed states (Delgado & Mielnik, 1998; Delgado, Mielnik & Reyes, 1998; Ma & Rhodes, 1989),

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to name but a few. It seems fair to say that two-level quantum systems have gained much attention owing to the high expectations for manipulating quantum information. The first physical realization of a qubit was carried out in 1995 using solid state systems (Cirac & Zoller, 1995). Since then, precise control of these systems for applications in quantum computing has become increasingly realistic due to theoretical and technological developments (Barenco, 1995; Barros, Stute, Northup, Russo, Schmidt & Blatt, 2009; Benhelm, Kirchmair, Roos & Blatt, 2008; Degani, Zanna, Saelen & Nepstad, 2009; Gisin, Ribordy, Tittel & Zbinden, 2002; Gu, Doherty & Nielsen, 2008; Häeffner, Roos & Blatt, 2008; Lee *et al.*, 2003; Loss & DiVincenzo, 1998; Maday & Turinici, 2003; Maurer, Becher, Russo, Eschner & Blatt, 2004; Nielsen & Chuang, 2000; Nielsen, Dowling, Gu & Doherty, 2006a, 2006b; Rabitz, de Vivie-Riedle, Motzkus & Kompa, 2000; Vandersypen, Breyta, Steffen, Yannoni, Sherwood & Chuang, 2001; Stievater *et al.*, 2001; Tian, Barber, Fischer & Babbitt, 2004; Vandersypen & Chuang, 2005; Wei & Nori, 2004).

Quantum computing promises to deliver more efficient ways to store and process information. This requires quantum algorithms and these, in turn, consist of sequences of quantum logic gates. Selective control protocols are the basis in the design of these processes.

Yet, in many theoretical and experimental approaches, selective manipulation by means of resonant driving fields has been *a priori* assumed. Although efforts have been made in the development of scientific software in the field of quantum computing (Tabakin & Juliá Díaz, 2011), it is still necessary to develop more applications to close the gap between theory and experiment. In this article we have as starting point a first principle model consisting in a two-level quantum system manipulated by a generic external field. An algebraic solution of the Pauli-Schrödinger equation is reviewed. Since such exact solutions are deeply sensitive to the parameters in the model we stressed that high accuracy programs are needed. The numerical implementation[1] of the model allows us to discuss some non-trivial processes in which the character of the control sequence assists the accumulation of non-resonant effects, triggering the loss of selectivity in the quantum algorithms.

As corollary aim in this paper is to include noise effects on qubit manipulations. Previous research such as Brańczyk, Mendonça, Gilchrist, Doherty & Bartlett (2007) addresses the control problem in two-level systems including a dephasing noise, that is, a phase flip is applied within a fixed $p$ probability, as in a Bernoulli trial. In Delgado (2010) some control procedures in bipartite systems are studied in the presence of a parasite field that is modelled through a homogenous field of variable lenght.

Along similar lines, the method we propose here can be interpreted as introducing a "signal noise" in the parameters of the model. Contrasting our results against experimental data from leading experimentalist groups (Stievater *et al.*, 2001; Lee *et al.*, 2003), we show that our simple noise model captures the observed behavior well.

## The model

It is widely accepted that the dynamics of a quantum system are defined by a unitary operator $U(t, t_0)$ describing the evolution in the time interval $[t_0, t]$. If the state of the system at an initial time $t_0$ is $|\psi(t_0)\rangle$, then, at an arbitrary moment $t$, it is given by

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle. \tag{1}$$

This operator fulfills the first order initial value problem ($\hbar = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0) = -\mathrm{i}H\mathrm{U}(t,t_0), \qquad U(t_0,t_0) = \mathbf{1}, \tag{2}$$

where $H$ is the Hamiltonian of the system.

The equation (2) for a time-dependent Hermitian Hamiltonian has solutions with exponential representation in terms of Hermitian operators $H_{\mathrm{ef}}(t, t_0)$ known as the *effective Hamiltonians* of the system in $[t_0, t]$

$$U(t,t_0) = \mathrm{e}^{-\mathrm{i}(t-t_0)H_{\mathrm{ef}}(t,t_0)}. \tag{3}$$

The point of departure for the quantum manipulation processes is the selection of an initially stationary system. We focus on the simplest case of a two level system (or qubit). This choice is sufficiently general in the sense that it allows to implement quantum control protocols that can be applied within the framework of the quantum information theory. Further extension of our results to an ensemble of qubits (the q-register) lies out of the scope of this article.

The qubit is represented by a (free) Hamiltonian $H_0$ whose eigenvalues $E_0$, $E_1$ and corresponding eigenstates $|0\rangle$, $|1\rangle$ are known. In the Hubbard representation we have

$$H_0 = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|. \tag{4}$$

[1] The code for the simulation was written by the authors in C++. All the graphics were obtained using the gnuplot utility.

The solution of (2) is straightforwardly constructed for the Hamiltonian $H_0$, namely

$$U_0(t,t_0) = e^{-i(t-t_0)H_0}. \tag{5}$$

Notice that $T_0 = 2\pi/\omega_0$, with $\omega_0 = E_1 - E_0$, is the period of a cyclic evolution process generated by $H_0$. In fact, in each time interval $[t_0, t_0 + T_0]$, the evolution operator $U_0(T_0) = e^{-iT_0H_0}$ is proportional to unity in the state space. It is not difficult to check that $e^{-iT_0E_0} = e^{-iT_0E_1}$, meaning that when applying $U_0(t_0 + T_0, t_0)$ to an arbitrary state

$$|\psi\rangle = a_1|1\rangle + a_0|0\rangle, \tag{6}$$

a new state is obtained that differs from the previous one only by a phase factor.

The main task in the control problem is to construct the proper Hamiltonian $H(t)$ which generates the unitary operator $U(t, t_0)$ transforming $|\psi(t_0)\rangle$ into $|\psi(t)\rangle$ consistently with (1)-(2). The most direct idea is to perturb a stationary system (the qubit) by using a time dependent external field

$$V(t) = \varepsilon_{01}(t)|0\rangle\langle 1| + \varepsilon_{10}(t)|1\rangle\langle 0|, \tag{7}$$

with $\varepsilon_{01}(t) = \bar{\varepsilon}_{10}(t)$ for $V(t)$ to be a Hermitian control operator (the bar stands for complex conjugation). Coefficient $\varepsilon_{01}$ is the transition amplitude between states $|0\rangle$, $|1\rangle$ and it is proportional to the field strength.

In the semiclassical approach the system interacts with a very soft, coherent, external field which does not cause radiative jumps. The time-dependent Hamiltonian is then $H(t) = H_0 + V(t)$.

In the resonant control technique, the dynamic operations are generated by exploiting the capability of fields of particular frequencies to induce transitions between two states of the qubit. In these operations the driving field is generally a harmonic, monochromatic external electromagnetic field. The control operator in the rotating wave approximation (RWA) has the form

$$V(t) = \frac{\varepsilon}{2}\left[e^{i\omega t}|0\rangle\langle 1| + e^{-i\omega t}|1\rangle\langle 0|\right], \tag{8}$$

where $\varepsilon$ stands for the field strength and $\omega$ for the field frequency. The matrix representation of $H(t)$ can be expressed in terms of spin Pauli matrices: Letting $E_1 = \omega_0/2$ and $E_0 = \omega_0/2$,

$$H(t) = \frac{\omega_0}{2}\sigma_z + \frac{\varepsilon}{2} e^{-\frac{i}{2}\omega t\sigma_z}\sigma_x e^{\frac{i}{2}\omega t\sigma_z}. \tag{9}$$

The solution of (2) for this case can be immediately written (Cruz y Cruz & Mielnik, 2007).

$$U(t,t_0) = e^{-\frac{i}{2}\omega(t-t_0)\sigma_z} e^{-\frac{i}{2}\left[\varepsilon_x\sigma_x + \varepsilon_y\sigma_y + (\omega_0-\omega)\sigma_z\right](t-t_0)}, \tag{10}$$

with $\varepsilon_x = \varepsilon\cos(\omega t_0)$ and $\varepsilon_y = \varepsilon\sin(\omega t_0)$.

The reader may observe that for $\omega = \omega_0$ a complete period of the external field coincides with one period of the free evolution of the system. This basic process and the external field are then "in phase" and the control effects are amplified. This suggests that selective control operations can be induced on qubits just by tuning the external field to the proper frequency. The effects of this evolution operator for different values of the external field parameters can be easily understood by considering the geometrical picture of the space of states given by the Bloch sphere[2] representation (Bengtsson & Zyczkowski, 2006; Fernández & Rosas-Ortiz, 1997; Hopf, 1931; Mielnik, 1968; Mosseri & Dandoloff, 2001; Ryder, 1980).

We could also study the effect of the operator (10) following the evolution of the affected state. The complex coefficients $a_0(t)$, $a_1(t)$ in (6) now depend on $t$, and thus we define $\mathcal{P}_{|i\rangle}$ as the probability of finding the system in state $|i\rangle$ ($i = 0,1$) in a measurement at time $t$

$$\mathcal{P}_{|0\rangle}(t) = |a_0(t)|^2, \qquad \mathcal{P}_{|1\rangle}(t) = |a_1(t)|^2. \tag{11}$$

Aditionally the restriction

$$\mathcal{P}_{|0\rangle}(t) + \mathcal{P}_{|1\rangle}(t) = 1, \tag{12}$$

gives us the equation of a sphere $S^3$ embedded in $\mathbf{R}^4$. This condition defines each state up to a general phase factor.

### The resonant q-control

Let us consider the evolution operator (10) from this point of view. If $\varepsilon = 0$, the (free) evolution operator

$$U_0(t,t_0) = e^{-\frac{i}{2}\omega_0(t-t_0)\sigma_z}, \tag{13}$$

causes rotations of $|\psi(t)\rangle$ around the $\mathbf{z}-$ axis with an angular frequency $\omega_0$. This is the natural evolution of the system which is present even if no external fields are applied (figure 1).

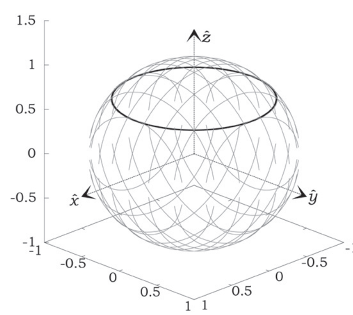


**Figure 1.** The free evolution of the qubit visualized on the Bloch sphere. It consists of circular trajectories. The constant value of $z$ indicates that the probabilities (11) remain the same during the evolution.
Source: Authors own elaboration.

[2] An explicit form of the Hopf map we chose: $x(t) = 2Re\left(a_0(t)\bar{a}_1(t)\right)$, $y(t) = 2Im\left(a_0(t)\bar{a}_1(t)\right)$, $z(t) = |a_1(t)|^2 - |a_0(t)|^2$.

When the external field $V(t)$ is applied, the evolution operator (10) consist of two factors. The first one is the unitary transformation to the rotating frame, while the second one includes information about the evolution in this frame. In the rotating frame, the state therefore performs circular trajectories around an axis $\mathbf{z}'$ defined by the unit vector

$$\mathbf{n} = \frac{1}{\Omega}\left(\varepsilon_x, \varepsilon_y, \omega_0 - \omega\right), \tag{14}$$

with angular frequency $\Omega = \sqrt{\varepsilon^2 + \left(\omega_0 - \omega\right)^2}$ [figure 2(a)]. In the laboratory frame, the $\mathbf{z}'-$ axis rotates around the $\mathbf{z}-$ axis with angular frequency $\omega$, meaning that the trajectory is confined to a belt of width $\delta = 2\theta$, where $\theta = \arctan[\varepsilon/(\omega-\omega_0)]$ is the angle between the axes $\mathbf{z}$ and $\mathbf{z}'$, around the free trajectory.

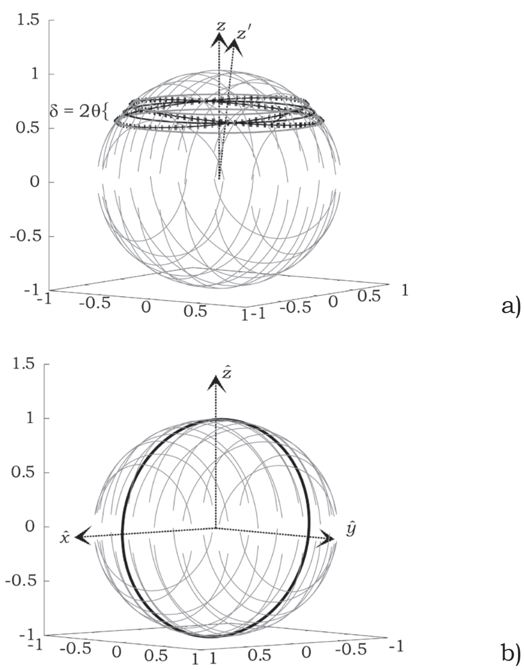

**Figure 2.** The Bloch sphere. (a) The free evolution of the qubit lies on a plane orthogonal to the $\mathbf{z}$ − axis (dashed curve), while the trajectory in the rotating frame when the driving field is applied lies on the plane orthogonal to $\mathbf{z}'$ (solid curve). (b) If the system performs Rabi rotations, its trajectory is a circle containing both orthogonal states $|1\rangle$, $|0\rangle$.
Source: Authors own elaboration.

In the non-resonant case ($\omega \neq \omega_0$), and for small enough values of the manipulating field strength ($\varepsilon/|\omega_0 - \omega| \ll 1$), the divergence $\theta$ between the $\mathbf{z}-$ and $\mathbf{z}'-$ axes will be so slight that the trajectory will almost coincide with that of free evolution.

In the resonant case $\omega = \omega_0$, the evolution operator is given by

$$U(T) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\varepsilon T\sigma_x}, \tag{15}$$

since in this case the free evolution term transforms into a phase factor. The qubit then performs an effective rotation around the $\mathbf{x}-$ axis on the Bloch sphere with angular frequency $\varepsilon$. This indicates that even in the case $\varepsilon \ll 1$, if this control operation is repeated many times, the system, in a particular state, can eventually reach the corresponding orthogonal one. These are called the Rabi rotations (Rabi, Ramsey & Schwinger, 1954) [figure 2(b)].

These facts lead to the assumption that a quantum system can be selectively manipulated just by means of resonant external fields. Yet, some care must be taken if one seeks to implement quantum control operations on a system which has multiple spectral levels or that is composed by a number of qubits. In that case, it is more convenient to apply sequences of external fields of specific frequencies, each one addressing a particular qubit (Khaneja & Glaser, 2001; Maurer *et al.*, 2004; Nigmatullin & Schirmer, 2009; Schirmer, Greentree, Ramakrishnaand & Rabitz, 2002; Schirmer, Pullen & Pemberton-Ross, 2008; Schirmer & Solomon, 2004; Vandersypen & Chuang, 2005; Vandersypen *et al.*, 2001; Wei & Nori, 2004), alternating with free evolution intervals. The general assumption is that each qubit will disregard the non-resonant part of the sequence.

Yet, for some cases the "on-off" sequences allow the accumulation of the non-resonant effects that are supposed to be negligible. For some particular protocols the errors become larger as the number of operations increases, in such a way that either the final state may be very different from the expected one, or even the qubit may be out of sight. The phenomenon is crucial for systems with short coherence time that must be driven by intense fields. This fact is significant for applications in quantum computing, where the precision must be high and the number of operations is large (*e.g.* an experimental Shor's algorithm may consist of approximately 300 pulses interrupted by time intervals of free evolution, applied to an ensemble of 7 qubits (Vandersypen *et al.*, 2001).

#### Testing a resonant manipulation of a single qubit

In the present section we analyze concrete realizations of resonant control protocols. A great number of experiments in the last decade has been implemented using solid state systems as qubits. As discussed previously, quantum control requires an external field to induce state transitions. In the resonant regime the field strength and interaction time are decisive to drive the system from the initial to the target state. From the pure mathematical point of view the tuning of these parameters is just a direct exercise using the geometrical picture of the Bloch sphere. Nevertheless, in the experimental realizations of control operations some practical imperfections should not to be ignored (the environmental noise, inaccuracies in the driving field parameters, to name but a few).

In order to observe the Rabi oscillations experimentally the chosen observable may be *e.g.*, the population of the first excited state. In our simulations, the occupation probability for the first excited state $\mathcal{P}_{|1\rangle}$ (11) is the control parameter that, assuming ergodicity, corresponds to the population. Our results are compared to those in (Lee *et al.*, 2003) in which the authors present a resonant technique to induce Rabi rotations on trapped-ion qubits via stimulated Raman transitions.

To reproduce the ideal resonant case we chose $\omega = \omega_0 = 2\pi \times 14.53\text{GHz}$.[3] The field strenght now plays the role of rotation frequency around the $\mathbf{x}'-$ axis. The value $\varepsilon = 2.25 \times 10^{-5}\omega_0$ was found numerically and calculated to produce seven Rabi rotations in a period of $200 \times 10^{-5}s$, as a cross check. In figure 3 we observe the variation of $\mathcal{P}_{|1\rangle}$ with the simulation time (that in this case corresponds to the pulse duration). This operation produces effectively a NOT gate in a period of $14 \times 10^{-5}s$. In the Bloch sphere the state goes from $|0\rangle$ to $|1\rangle$ within this time interval as it is shown in figure 4.

In order to account for the deviation from this ideal situation, we introduced small modifications in the field frequency. We considered two small variations in the value of the field frequency, namely $|\Delta\omega| = 1 \times 10^{-6}\omega_0$ and $1 \times 10^{-5}\omega_0$ where $\Delta\omega = \omega - \omega_0$ (see figure 5). The net effect of varying the field frequency is to reduce the amplitude of the oscillations. The detuning $\Delta\omega$ from the resonant frequency deviates the rotation axis (14) of the state in the Bloch sphere. The entire trajectory will then be modified in such a way that it cannot reach the antipodal point.

Note that in figure 5, for $\Delta\omega > \omega_0 \times 10^{-5}$ the amplitude of the oscillations is practically negligible, meaning that for this detuning the driven field can be considered as non resonant. This fact allows to determine qualitatively to which extent a frequency field can be considered as resonant to the system (a threshold in the parameters).

Distinctively, a slight variation to the field strength modifies only the Rabi frequency. The system will be able to perform complete Rabi rotations that may be advanced or retarded depending on the field strength, *i.e.*, for weaker/stronger fields it takes more/less time for the system to reach the antipodal state.

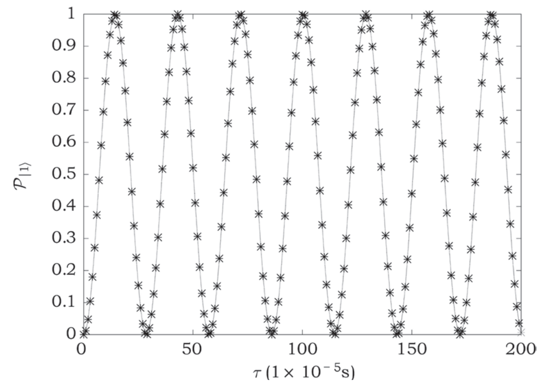


**Figure 3.** The evolution of the probability (11) of finding the system in state $|1\rangle$ for the resonant case. The initial state is $|0\rangle$. After a lapse of time equivalent to 196 basic periods of $1 \times 10^{-5}s$ we observe 7 Rabi oscillations. The curve is drawn as a reference to guide the eye.
Source: Authors own elaboration.

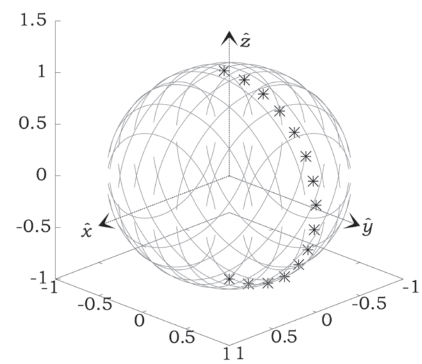


**Figure 4.** An effective NOT gate. The trajectory on the Bloch sphere shows the system going from $|1\rangle$ to $|0\rangle$.
Source: Authors own elaboration.

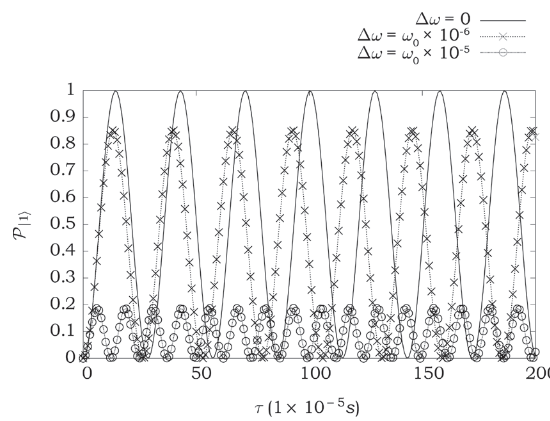


**Figure 5.** The evolution of $\mathcal{P}_{|1\rangle}$ for 3 different frequencies. In all simulations the initial state is taken to be $|0\rangle$. As in Figure 3, the solid curve that corresponds to the resonant case is included for reference. For $\Delta\omega = \omega_0 \times 10^{-6}$, $\omega_0 \times 10^{-5}$ we observe that the net effect is to reduce the amplitude of the oscillations, as well as to decrease their frequency.
Source: Authors own elaboration.

[3] Since the axis of rotation (14) is defined in terms of the frequencies and the field strength, it is convenient to express these quantities in units of one of them; for this purpose, we selected $\omega_0$.

In figure 6 we present the effect of a small variation $\Delta\varepsilon$ in the control field strength, taking $\Delta\varepsilon = \pm(1/6)\varepsilon$. As can be observed, these variations do not affect the amplitude of the oscillations; instead, they induce a drift in the Rabi frequency. However, note that the effect of averaging over different processes is to modulate the amplitude of the oscillations. As in the experimental regime, the Rabi flopping plots are the result of the averaging over an ensemble of experiments (see Stievater *et al.*, 2001; Lee *et al.*, 2003). Identifying the origin of the enveloping decay becomes complicated. Yet, in the simulation, the effects of variations in strength and frequency of the driving field are essentially different.

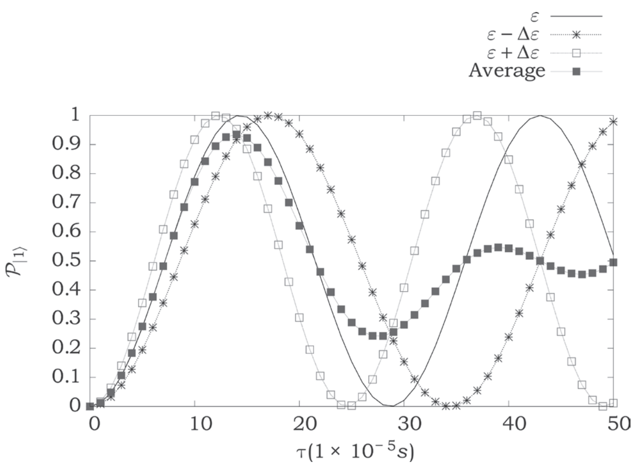


**Figure 6.** The evolution of $\mathcal{P}_{|1\rangle}$ for 3 different field strengths. In all simulations the initial state is taken as $|0\rangle$. The solid curve corresponds to $\varepsilon = 2.25 \times 10^{-5}\,\omega_0$. For $\varepsilon \pm \Delta\omega$ we can observe a drift in the Rabi frequency. Finally we present the average between the last two cases.
Source: Authors own elaboration.

**Non resonant control operations**

Since the external field is tuned in a different frequency to that of the qubit, the goal in non resonant control operations is to mantain the system in the initial state, i.e. the system has to remain stable. In the following, a single qubit of natural frequency $\omega_0$ in the presence of some sequences of non resonant external fields is considered. Different cases are explored and numerical results are reported. These may shed some light in the design of precise quantum computing algorithms.

Our tests are based on a finite sequence of "on-off" stages in the control field. The choice of such a process is mainly due to two reasons: firstly, it is sufficiently general and it allows us to generate several control protocols; secondly, it emulates the clock cycles scheme in classical computation.

A similar methodology in modeling the distortion in a bipartite system was implemented in (Delgado, 2010). However, the main difference with our study is related to the control protocol. The distortion occurs in the length of the single cycle of interaction with the instrument control, which for us would correspond to one clock cycle, making it impossible to analyze the effect of a cumulative error in the sequences of interaction between the system an the external field.

In this context, assuming that a control field of frequency $\omega$ is applied during a period $T_c$ followed by a lapse $T_f$ of free evolution. The evolution operator of the system in a complete period of time $T = T_c + T_f$ then reads (Cruz y Cruz & Mielnik, 2007; Cruz & Medina, 2010).

$$U(T) = e^{-i\frac{1}{2}\left[\omega_0 T_f + \omega T_c\right]\sigma_z} e^{-i\frac{1}{2}T_c\left[\left(\omega_0 - \omega\right)\sigma_z + \varepsilon\sigma_x\right]}. \qquad (16)$$

In a realistic simulation one has to consider that the physical instrument that generates the external field is not perfect. We stress that the theoretical treatment reviewed in this paper is for a generic system. Under the appropiate translation of the constants involved in (16) it can represent the previously described evolution of a concrete physical realization of a qubit-field coupling system. For instance we could consider a solid state qubit (ions or atoms in which only two energy levels are considered) having a laser beam acting as a control tool. Since the spectral emission line from which the beam originates does have a finite width, even the most accurate monocromatic laser must consists of the superposition of several frequencies. We propose that this effect can be captured considering a single effective frequency that deviates from an ideal mean value. This can be achieved introducing a "noise signal" into the field parameters (Lathi, 1995). The frequency of the external field is now allowed to take random values inside a Gaussian probability distribution with mean at a non resonant value. For $|\omega_0 - \omega| \ll 1$ the dynamics are confined to a narrow region of the Bloch sphere, again resembling the stationary non resonant trajectories [figure 7(b)]. Yet, for $(\omega/\omega_0) \sim 1$, the information about the state of the system may be rapidly lost in those protocols. Figures 7-8 show some points of the trajectories for $\omega = 2\pi \times 470$ MHz with field strength $\varepsilon = 0.01\omega_0$ and $\varepsilon = 0.1\omega_0$ respectively, and frequencies in the Gaussian distribution with mean at $\omega = 2\pi \times 202$ MHz and standard deviation $\sigma = \omega_0/100$. They reveal that the non resonant effects cannot be neglected for a great number of control operations. The complete sequence can lead to radically different results. In figure 7 we illustrate two trajectories using two different sequences of pulses in the same Gaussian distribution of frequencies, for a field strength $\varepsilon = 0.01\omega_0$ In 7(a) the state of the system is confined to a narrow region around the free evolution trajectory. In contrast, the trajectory in 7(b) is spread in the upper hemisphere $z > 0$. Even more serious is the case of strong field $\varepsilon = 0.1\omega_0$ (figure 8), since it may drive the system completely out of control. The points of the trajectory can be seen spreading all over the sphere after a relatively large sequence of 100 steps.

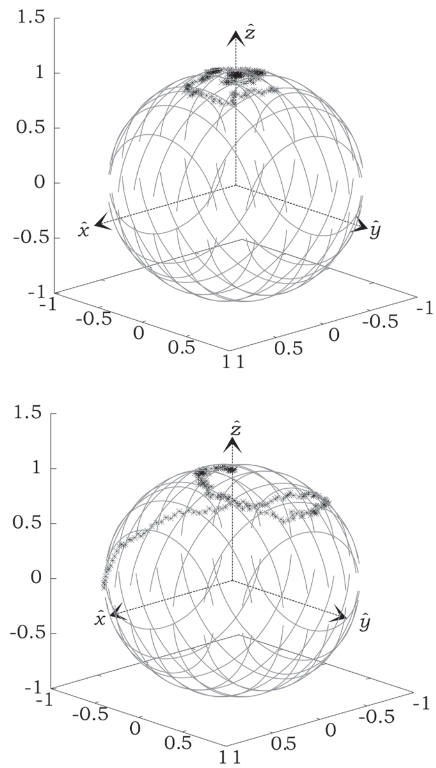


**Figure 7.** Two trajectories of the state considering the same physical conditions. $\omega_0 = 2\pi \times 470$ MHz under the action of external pulses of non resonant frequencies taking values in the Gaussian distribution with mean at $\omega_0 = 2\pi \times 202$ MHz and $\sigma = 2\pi \times 4.7$ MHz. The field strength is $(\varepsilon/\omega_0) = 0.01$. For these processes $T_c = 1.65 \times 10^{-9}s$, $T_f = 0.49 \times 10^{-9}s$ and $a_1(t_0) = 1$, and 200 pulses were applied.
Source: Authors own elaboration.

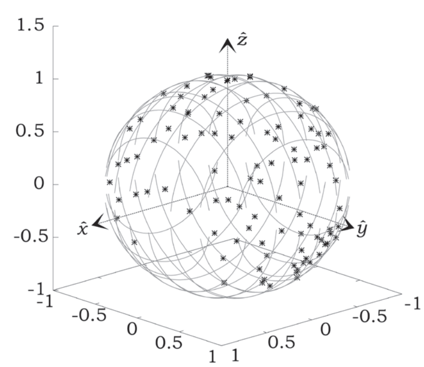


**Figure 8.** Trajectory of the state in the strong field case $\varepsilon = 0.1\omega$ for a 100-pulse sequence. Other parameters remain the same as those considered in figure 7.
Source: Authors own elaboration.

Now consider the case in which fields of 5 different frequencies are applied randomly chosen from a uniform probability distribution. Field pulses are separated by free evolution time intervals. This process emulates a system of 6 non-interacting qubits, 5 of them with the chosen 5 different natural frequencies and one of them characterized by $\omega_0$ as its natural oscillation frequency. The external field is selected to be resonant in each step to one of the first mentioned 5 qubits. Since $\omega \neq \omega_0$, what would be expected is that the qubit at the center of our analysis may stay stable in its initial state.

This is not the situation in figure 9(a) that shows the first 100 points of the trajectory for a qubit with frequency $\omega_0 = 2\pi \times 470$ MHz. This number of steps is necessary for the state initially in $|0\rangle$ to reach state $|1\rangle$. In this case, fields of strength $\varepsilon = 0.1\omega_0$ and frequencies $\omega/(2\pi) = 51$MHz, 77MHz, 126MHz, 202MHz and 500MHz (Vandersypen & Chuang, 2005) were distributed during equal "on-off" time intervals of length $\pi/\omega_0$. Figure 9(b) shows some points of the trajectory for frequencies of the field pulses coinciding with $\omega_0$.

Since the number of steps was 100, the resonant field was applied, on average, enough times for the qubit to perform a complete Rabi rotation. The picture shows that the final state is slightly deviated from the initial one, meaning that the presence of the non resonant fields does not only delay the process, but it also modifies the expected trajectory.

## CONCLUDING REMARKS

There are still certain gaps in the existing manipulating techniques so far disregarding the problem of selectivity breaking in quantum algorithms. Resonant control operations seem to be the most efficient way to manipulate a quantum system; however, care should be taken with non resonant perturbations in complex systems driven by long sequences of external fields. The numerical results reported above manifest that the system can be destabilized by the effect of interrupting the control operations. The situation becomes even more dramatic when uncertainties in the manipulating instrument (arising *e.g.* from finite precision) or noise are considered. There is, however, some evidence that the non resonant fields do not always destabilize the system, making possible to drive it efficiently.

Noise effects can be modeled considering slight variations in the parameters of our model. The proposed methodology is similar to that used to include noise in signal processing.

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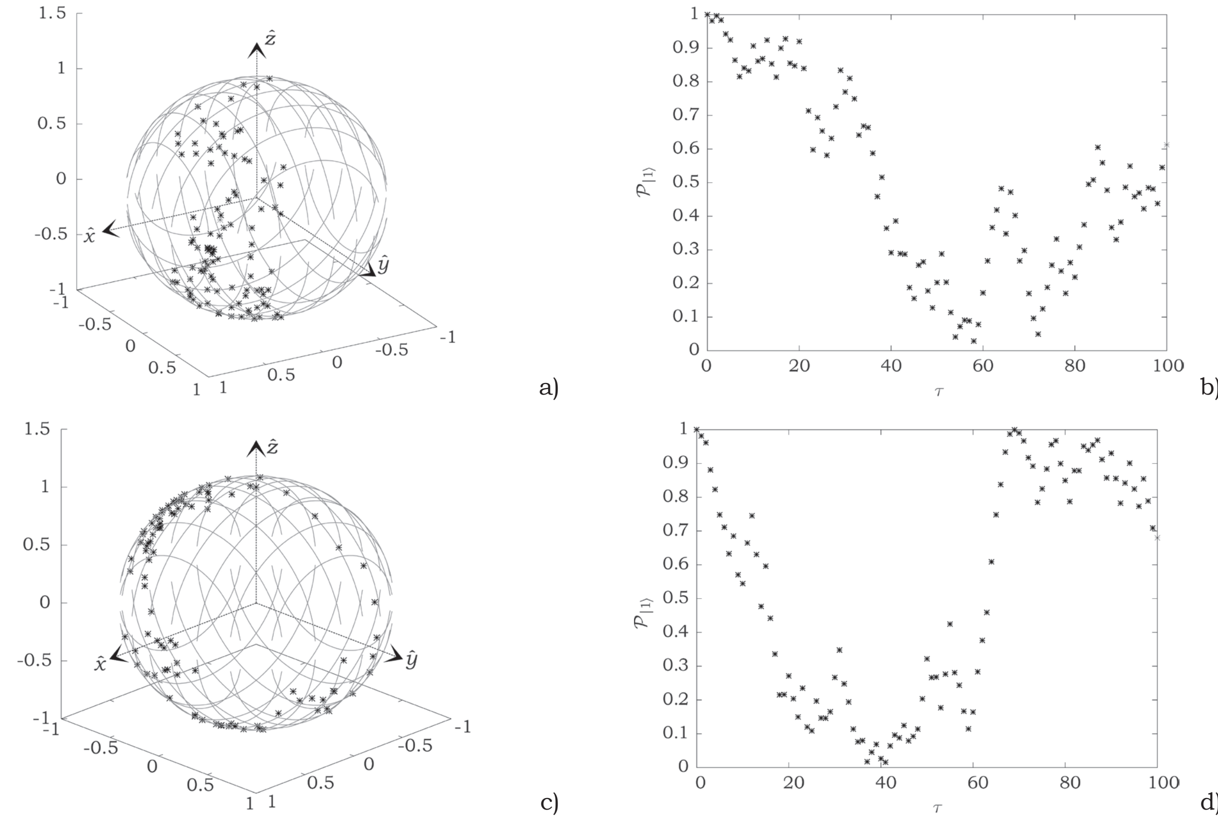

**Figure 9.** Some points of the trajectories in the Bloch sphere of the qubit $\omega_0 = 2\pi \times 470$MHz manipulated by fields of 5 different frequencies. In (b) and (d) the evolution of the probability defined in (11) is presented. In this case $T_c = T_f = \pi/\omega_0 = 1.06 \times 10^{-9}s$ and $(\varepsilon/\omega_0) = 0.1$. (a)-(b) For $(\omega/2\pi) = 77$ MHz, $500$MHz, $126$MHz, $-51$MHz and $202$MHz; In (b) we show the corresponding projection of the previous trajectory in the $F-$ axis; (c)-(d) for $(\omega/2\pi) = 470$MHz, $500$MHz, $126$MHz, $-51$MHz and $202$MHz, and its corresponding projection.
Source: Authors own elaboration.

The simulation of the proposed model makes it possible to explore the stability of control protocols. Thus, it is possible to define thresholds of tolerance in the model parameters. A proposal of a fine control mechanism such as that presented by Brańczyk and Delgado is outside the remit of this study. The correction mechanism needs the prediction of the most likely final state. Based on the methodology presented here such a requirement could be explored numerically using Monte Carlo simulations. A survey in that direction is in progress.

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